On a Complete System of Invariants of Two Triangles.

by

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Introduction.

The simpler invariants of two triangles as derived from the common set up by them, when regarded as two cubic curves of the third order and of the third class respectively, have been discussed at length by Dr. Hun.* This system there derived is clearly incomplete as is pointed out in the article. The present Note seeks to present a complete system, by means of which all invariant relations can be conveniently expressed. Some [illegible]

*Transactions American Math. Soc. Vol. [illegible], No. 1. p. 39 ff

is also given to a few of the covariants but no exhaustive treatment has been attempted.

In developing the system of invariants proposed in the present paper the element was from [illegible], simplified [illegible] to the [illegible] [illegible]. In fact the fundamental problem which to a large extent gave rise to this paper was one proposed by Prof. Morley, viz.: What is the invariant relation for two triangles so that a conic may be drawn on the vertices of one triangle and touch the sides of the other? This problem will be discussed in a subsequent chapter. In this way the selection of the present system of invariants was largely a matter of experiment

and observation, the object of course being to select such as would enable one to write down invariant relations in the most simple form, and at the same time, to have the fundamental invariants themselves as easily interpreted geometrically as possible. In the original solution of these special problems one triangle was invariably taken as the reference triangle and the invariant relation was consequently expressed in terms of the coefficients of the other. Naturally we avail ourselves of the same simplification in much of the discussion following.

It has, however, seemed advisable in this paper to follow what may be termed the historical, and at the same time more logical order, [illegible]

begin by giving the movements in their most general form. These were obtained in the first place from the simplified forms very readily as soon as the latter and their interpretation were known.

§1. The Primary System.

Take the one triangle as a 3-line in the most general symbolic form

$$(\alpha x)\,(\beta x)\,(\gamma x) = 0\,,$$

and the other as a 3-point

$$(a\xi)\,(b\xi)\,(c\xi) = 0$$

where the expressions in these are the ordinary symbolic products, i.e.

$$(\alpha x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\,, \text{ etc.}$$

The system of invariants which we will use and later prove to be a complete system is the following:—

1) $$\Delta_1 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$2)\quad D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

$$3)\quad I_1 = (a\alpha)(b\beta)(c\gamma) + (a\alpha)(b\gamma)(c\beta) + (b\alpha)(c\beta)(a\gamma) + (b\alpha)(c\gamma)(a\beta) + (c\alpha)(a\beta)(b\gamma) + (c\alpha)(a\gamma)(b\beta).$$

$$4)\quad I_2 = (a\alpha)(a\beta)(b\beta)(b\gamma)(c\gamma)(c\alpha) + (a\gamma)(a\alpha)(b\alpha)(b\beta)(c\beta)(c\gamma) + (a\beta)(a\gamma)(b\gamma)(b\alpha)(c\alpha)(c\beta) + (a\alpha)(a\beta)(b\gamma)(b\alpha)(c\beta)(c\gamma) + (a\gamma)(a\alpha)(b\beta)(b\gamma)(c\alpha)(c\beta) + (a\beta)(a\gamma)(b\alpha)(b\beta)(c\gamma)(c\alpha).$$

$$5).\quad D_2 = (a\alpha)(a\beta)(b\beta)(b\gamma)(c\gamma)(c\alpha) + (a\gamma)(a\alpha)(b\alpha)(b\beta)(c\beta)(c\gamma) + (a\beta)(a\gamma)(b\gamma)(b\alpha)(c\alpha)(c\beta) - (a\alpha)(a\beta)(b\gamma)(b\alpha)(c\beta)(c\gamma) - (a\gamma)(a\alpha)(b\beta)(b\gamma)(c\alpha)(c\beta) - (a\beta)(a\gamma)(b\alpha)(b\beta)(c\gamma)(c\alpha).$$

$$6)\quad I_3 = (a\alpha)(a\beta)(a\gamma)(b\alpha)(b\beta)(b\gamma)(c\alpha)(c\beta)(c\gamma).$$

D_2 can be written as a [illegible]

concomitant form to [illegible] as

a determinant

$$D_x = \begin{vmatrix} (a\alpha)(a\beta) & (a\beta)(a\gamma) & (a\gamma)(a\alpha) \\ (b\alpha)(b\beta) & (b\beta)(b\gamma) & (b\gamma)(b\alpha) \\ (c\alpha)(c\beta) & (c\beta)(c\gamma) & (c\gamma)(c\alpha) \end{vmatrix},$$

and I_2 is the same determinant expanded and all signs made positive.

Following Sir [illegible]'s notation*
for indicating the collective degree
in Roman and Greek letters
respectively, the concomitants are represented as follows:-

$D_1(3,0)$; $\Delta_1(0,3)$; $D_2(6,6)$;
$I_1(3,3)$; $I_2(6,6)$; $I_3(1,1)$.

It is just as well to employ a
similar notation to indicate the
degree in the actual coefficients
of the 3-point and 3-line respec-

* Ibid. p. 42.

ving. We then have

$D_1(1,0)$; $\Delta_1(0,1)$; $D_2(2,2)$;

$I_1(1,1)$; $I_2(2,2)$; $I_3(3,3)$.

The only advantage of this form is that it gives the reason for the choice of subscripts attached to our six fundamental invariants.

From these 6 independent invariants we would expect to be able to form 4 independent absolute invariants, and such is found to be the case. Since an absolute invariant must be of degree zero in both Greek and Roman letters, the 4 independent ones, (i.e. a set of 4) can be built up easily from the scheme above. Probably the simplest set is,-

$$\frac{I_2}{D_2} ; \quad \frac{I_1}{D_1 \Delta_1} ; \quad \frac{I_1^2}{I_2} ; \quad \frac{I_1 I_2}{I_3} .$$

That any other form of given de-

upon them be built up forms [illegible] is seen [illegible] readily [illegible] [illegible].

[illegible]

$$\frac{I.\Delta.[illegible]}{T} [illegible] \frac{I}{I_2} \cdot \frac{I_1}{I} \cdot \frac{[illegible]}{I_1}$$

$$\frac{[illegible]^2}{[illegible]} = \frac{I_2}{I_2} \cdot \frac{I_1^2}{I_2}, \text{ and so on.}$$

This result was to be expected [illegible] the simple meaning of constant [illegible] as that from triangles have [illegible] 4 absolute [illegible].

It is observed that the D's are determinants hence change sign when two letters are interchanged. Of course when equated to zero this makes no difference. On the other hand this fact enables one at a glance to tell whether the D's enter to an odd or even power in any given invariant form.

§ 2. Simplified Forms of the Fundamental Invariants.

If the 3-line be now taken as the reference triangle, all the invariants of the foregoing system except Δ_1, become homogeneous, rational, integral functions of the Roman letters, i.e., of the coefficients of the 3-point.* Δ_1 assumes the value unity. The simplification of the forms of the invariants as given in the following paragraph is a very easy matter, for the coefficients of the type $a_2 b_3 c_1$ and $\alpha_2 \beta_3 \gamma_1$ are zero. So the only combination of Greek letters we need consider is $\alpha_1 \beta_2 \gamma_3$ and

* Dr. Cobb, in an unpublished article, suggested a different complete system, based almost exclusively on this fact.

[illegible] by [illegible] ap-
pear. Dropping these out the fun-
damental invariants assume the
very simple forms

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$I_1 = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 + a_1 b_3 c_2 + a_3 b_2 c_1 + a_2 b_1 c_3.$$

$$D_2 = \begin{vmatrix} a_2 a_3 & a_3 a_1 & a_1 a_2 \\ b_2 b_3 & b_3 b_1 & b_1 b_2 \\ c_2 c_3 & c_3 c_1 & c_1 c_2 \end{vmatrix}$$

$$I_2 = a_2 a_3 b_3 b_1 c_1 c_2 + a_1 a_2 b_2 b_3 c_3 c_1 + a_3 a_1 b_1 b_2 c_2 c_3 + a_2 a_3 b_1 b_2 c_3 c_1 + a_1 a_2 b_3 b_1 c_2 c_3 + a_3 a_1 b_2 b_3 c_1 c_2.$$

$$I_3 = a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3$$

The fact that I_1 and I_2 are merely D_1 and D_2 expanded and all signs made plus is of great advantage in remembering these forms.

§3. Proof of the Completeness of the System.

It is obviously of prime importance to prove that the system as given is complete, i.e. that all invariant relations between the two triangles can be expressed in terms of the system proposed in the preceding paragraphs.

Take the 3-line as the reference triangle and call it T_1. The 3-point, $(a\xi)(b\xi)(c\xi)=0$, we will call T_2. If we took the 3-line generally, $(\alpha x)(\beta x)(\gamma x)=0$ we would define a rational, integral invariant of the two triangles as a rational, integral function of the coefficients which has the invariant property and is unaltered by a permutation of the 3 lines or of

the 3 points, that is, by any permutation of α, β, γ, or of a, b, c. So when the 3 lines are taken as the reference triangle, all rational simultaneous invariants, except Δ, (the determinant of the 3 lines which reduces to unity, but must be noted) of T_1 and T_2 are homogeneous rational integral functions of a_i, b_i, c_i, which are unaltered by any permutation of the letters or of the subscripts. The converse is equally true and any homogeneous, rational, integral function of the coefficients a_i, b_i, c_i, which is unaltered by a permutation of the letters or of the subscripts, has the invariant property. Hence any such invariant can be written down in the most general form in this summation;—

$$I = \sum a_1^{i_1} a_2^{i_2} a_3^{i_3} b_1^{j_1} b_2^{j_2} b_3^{j_3} c_1^{k_1} c_2^{k_2} c_3^{k_3}$$

where the exponents are subject to the following conditions:

$$i_1 + i_2 + i_3 = n \qquad i_1 + j_1 + k_1 = n$$

$$j_1 + j_2 + j_3 = n \qquad i_2 + j_2 + k_2 = n$$

$$k_1 + k_2 + k_3 = n \qquad i_3 + j_3 + k_3 = n$$

The necessity of these conditions is evident.

We next prove the lemma;- Every individual term of I, any invariant when I is of the nth degree in the coefficients of the n-point, is the product of n factors of the type $a_r b_s c_t$ where $r \neq s \neq t$.

To aid in establishing this lemma let

$$a_1 b_2 c_3 = \alpha_0 \qquad a_1 b_3 c_2 = \beta_0$$

$$a_2 b_3 c_1 = \alpha_1 \qquad a_2 b_1 c_3 = \beta_1$$

$$a_3 b_1 c_2 = \alpha_2 \qquad a_3 b_2 c_1 = \beta_2$$

[illegible] every letter with every possible sub-

script we can continue taking factors from it of type $a_i b_k c_l$, $i \neq k \neq l$. But some exponents may become zero, and others not and we must look at that case. We will suppose some exponent say $i_1 \neq 0$.
Then ~~from~~ our equations of condition become or tell us

$$i_1 > 0 \;\therefore\; i_2 < m \qquad i_3 < n \qquad \text{for } \Sigma i_i = n$$

also $j_2 + k_2$ [illegible] $j_3 + k_3$ [illegible]

$$\text{and} \quad j_1 < n \qquad k_1 < m$$

Hence $j_2 + j_3 > 0$ or j_1 would $= m$
and $k_2 + k_3 > 0$ or k_1 would $= m$

From these inequalities it follows
if $j_2 = 0$, $k_2 > 0$ & $j_3 > 0$ and since $i_1 > 0$
we have $a_1 b_3 c_2 = \beta_0$ as a factor.
If $k_3 = 0$, $j_3 > 0$, $k_2 > 0$, and again
[illegible] $a_1 b_3 c_2$ [illegible] as a factor.
[illegible] both j_2 and $k_3 = 0$, then $j_3 > 0$, $k_2 > 0$
& again $a_1 b_3 c_2 = \beta_0$ as a factor.
If $j_2 = 0$, or $k_3 = 0$, or both [illegible]

at that degree [illegible] $\alpha_1 \beta_1 \gamma_2 \dots = \lambda_0$ would be a factor. Taking out this factor α_i or β_i, we have left a factor of same type as before of degree $n-1$ instead of n. We can continue factoring out an α_i or a β_i, until the whole invariant form is expressed in terms of α_i and β_i. This proves our lemma and we can write

$$I = \cdot \sum \alpha_0^i \alpha_1^j \alpha_2^k \beta_0^l \beta_1^m \beta_2^n \, .$$

Now since I must remain unaltered by all permutations of a, b, c and of the subscripts $1, 2, 3$, it must remain unaltered by all permutations of the Greek letters and of their subscripts. To establish this; if we express the permutations as cycles in the case of the English letters it is an easy matter to express the corresponding permutation of the Greek letters as cycles [illegible].

If $(a\,b\,c)$ represents a cyclic permutation we have

(1). $(a\,b\,c)$ is equivalent to $(\alpha_0\alpha_2\alpha_1)(\beta_0\beta_1\beta_2)$

(2). $(1\,2\,3)$ " " " $(\alpha_0\alpha_1\alpha_2)(\beta_0\beta_1\beta_2)$

(3). $(b\,c)$ " " " $(\alpha_0\beta_0)(\alpha_1\beta_1)(\alpha_2\beta_2)$

(4). $(2\,3)$ " " " $(\alpha_0\beta_0)(\alpha_1\beta_2)(\alpha_2\beta_1)$.

(5). $(a\,b)$ " " " $(\alpha_0\beta_1)(\alpha_1\beta_2)(\alpha_2\beta_0)$

This includes all independent types.

This discussion can be still further abbreviated by writing

$$\alpha_0+\alpha_1+\alpha_2 = r, \qquad \beta_0+\beta_1+\beta_2 = s,$$

$$\alpha_1\alpha_2+\alpha_2\alpha_0+\alpha_0\alpha_1 = r_2 \qquad \beta_1\beta_2+\beta_2\beta_0+\beta_0\beta_1 = s_2$$

$$\alpha_0\alpha_1\alpha_2 = \beta_0\beta_1\beta_2 = p_3$$

$$(\alpha_1-\alpha_2)(\alpha_2-\alpha_0)(\alpha_0-\alpha_1) = r \quad (\beta_1-\beta_2)(\beta_2-\beta_0)(\beta_0-\beta_1) = s.$$

Since any invariant I admits of the permutation $(\alpha_0\alpha_1\alpha_2)$ and contains a term $\alpha_0^i\alpha_1^j\alpha_2^k\beta_0^l\beta_1^m\beta_2^n$, it must contain the 3 terms obtained by cyclically advancing the α_i's. Since it also permits the permutation $(\beta_0\beta_1\beta_2)$, it must

contain the terms obtained by cycli-cally advancing the β_i's. In short, it must contain the 9 terms

$$\sum_{(\beta_0\beta_1\beta_2)} \beta_0^{l}\beta_1^{m}\beta_2^{n} \sum_{(\alpha_0\alpha_1\alpha_2)} \alpha_0^{i}\alpha_1^{j}\alpha_2^{k}\,.$$

If $i=j=k$ or $l=m=n$, these 9 terms reduce to 3, and if both equalities exist simultaneously, they reduce to a single term. In any event we have the product of two alternating functions of $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$ respectively. Carrying out the same argument for the other terms of I, we find that

$$\therefore\ I = R(r_1, s_1, r_2, s_2, p_3, r, s)$$

where R is a rational, integral, isobaric function of its arguments. The subscripts indicate the weight, r and s being of weight 3.

As a next step, r and s may be assumed to occur to the first degree only, for r^2 and s^2 can both be

expressed rationally in terms of the other 5 quantities. In fact we have

(6) $r^2 = r_1^2 r_2^2 - 4r_1^3 p_3 + 18 r_1 r_2 p_3 - 4r_2^3 - 27 p_3^2$. *

$s^2 = s_1^2 s_2^2 - 4s_1^3 p_3 + 18 s_1 s_2 p_3 - 4 s_2^3 - 27 p_3^2$.

We also notice that the permutation $(\alpha_1 \alpha_2)(\beta_1 \beta_2)$ changes the sign of r and s. Hence r and s can occur only in the combination rs.

Therefore we can now write

$$I = R_1(r_1, s_1, r_2, s_2, p_3) + rs \cdot R_2(r_1, s_1, r_2, s_2, p_3)$$

Again, the permutation $(\alpha_0\beta_0)(\alpha_1\beta_1)(\alpha_2\beta_2)$ interchanges r_i and s_i. Therefore I is unaltered by interchanging r_i and s_i. Or

(7) $R_1(r_1, s_1, r_2, s_2, p_3) = R_1(s_1, r_1, s_2, r_2, p_3)$.

Now take a term of R_1 as $r_1^i s_1^k r_2^l s_2^m p_3^n$, then from (7), R_1 must contain the sum of the 2 terms

$$p_3^n [r_1 s_1]^k [r_2 s_2]^m [r_1^a r_2^b + s_1^a s_2^b] = A_1, \text{ say,}$$

where for convenience we suppose $k \leq i$, $m \leq l$, $i - k = a$, $l - m = b$.

* Salmon – Leçons D'Algèbre Superieure. p 89–90.

To reduce still further, let

$$p_3^{n}[r_1 s_1]^{k}[r_2 s_2]^{m}[r_1^{a} s_2^{b} + s_1^{a} r_2^{b}] = A_2$$

Then combining A_1 and A_2

$$A_1 + A_2 = p_3^{n}[r_1 s_1]^{k}[r_2 s_2]^{m}(r_1^{a}+s_1^{a})(r_2^{b}+s_2^{b})$$

$$= S_1[r_1+s_1,\ r_2+s_2,\ r_1 s_1,\ r_2 s_2,\ p_3]$$

$$A_1 - A_2 = p_3^{n}[r_1 s_1]^{k}[r_2 s_2]^{m}(r_1^{a}-s_1^{a})(r_2^{a}-s_2^{a})$$

$$= (r_1-s_1)(r_2-s_2)\, S_2[r_1+s_1,\ r_2+s_2,\ r_1 s_1,\ r_2 s_2,\ p_3].$$

where S_i is a rational function of the arguments. This enables one to solve for A_1 in terms of $(r_1-s_1)(r_2-s_2)$ and the 5 arguments of S_i. Carrying out the same process for all terms of R_1 we find

$$R_1 = M_1[r_1+s_1,\ r_2+s_2,\ r_1 s_1,\ r_2 s_2,\ p_3]$$
$$+ (r_1-s_1)(r_2-s_2)\, M_2[r_1+s_1,\ r_2+s_2,\ r_1 s_1,\ r_2 s_2,\ p_3]$$

where M_i is a rational, integral, isobaric function of its arguments. R_2 can be expressed in the same way. Hence

Any rational, simultaneous, invariant of T_1 and T_2 is a rational, integral, isobaric function of $r_1+s_1,\ r_2+s_2,\ r_1 s_1,\ r_2 s_2,$

p_3, $(r_1 - s_1)(r_2 - s_2)$, and rs. Now these seven arguments are themselves invariants, hence we have as a final theorem:-

The following seven invariants form a complete system of the simultaneous, rational invariants of T_1 and T_2:-

$\underline{I_1 = r_1 + s_1}$, $\qquad \underline{J_2 = r_1 s_1}$,

$\underline{I_2 = r_2 + s_2}$, $\qquad \underline{J_3 = (r_1 - s_1)(r_2 - s_2)}$;

$\underline{I_3 = p_3}$, $\qquad \underline{I_6 = rs}$.

$\underline{I_4 = r_2 s_2}$,

Of these seven, the last two are skew. It does not follow from the argument presented that these seven are absolutely independent. In fact there is a syzygy connecting them, for, since we have already shown that r^2 and s^2 individually can be expressed in terms of the first six, it follows that I_6^2 can be so expressed. Hence the

first six constitute a complete system.

The system of seven arguments, which we have proven complete, was first suggested by Dr. Coble as one which might be suitable for the simple representation of all invariant relations between two triangles. A considerable amount of working with various possible combinations of these forms, which could be used as a fundamental or primary system, caused the writer to select the system presented in the body of this article as the simplest system available. This system is easily shown to be as good as the Coble system which has been proven complete.

We have finally chosen

$\underline{I_1 = r_1 + s_1}$, $\quad$ $\underline{D_1 = r_1 - s_1}$,

$\underline{I_2 = r_2 + s_2}$, $\quad$ $\underline{D_2 = r_2 - s_2}$,

$I_3 = p_3$, as a primary system.

Clearly the whole system can be expressed in terms of these five by substituting the former in the latter:

$$I_1' = I_1 \qquad I_2' = \tfrac{1}{4}(I_1^2 - D_1^2)$$

$$I_2' = I_2 \qquad I_3' = I_1 I_2$$

$$I_3' = I_3 \qquad I_4' = \tfrac{1}{4}(I_2^2 - D_2^2)$$

and of course I_6^2 can be expressed <u>rationally</u> although the form is somewhat long. Of course Δ_{11}, which is unity when the triangle is taken as the reference triangle (as in this paragraph) must be included in the complete system. Hence we have proven the fundamental theorem of this paragraph, viz.,– <u>The system of invariants proposed in the previous paragraph is complete; although for rational completeness we must add I_6 which has been defined above.</u>

The principles employed in this paragraph to prove the completeness

of the system, (i.e. if any invariant contains a term $\alpha_0^i \alpha_1^j \alpha_2^k \beta_0^l \beta_1^m \beta_2^n$, it will contain all terms obtained by permuting the α's and β's) will be found very useful in determining the invariant relations, as we need only consider about a single term of each type. This reduces the necessary algebra very materially. In some cases it is convenient to use the symbols [illegible] and [illegible] as defined in this section, but in general, being unsymmetrical, their use leads to complications.

It may be well to add the form of I_6 in terms of the coefficients of the three points:—

$$I_6 = (a_2 b_3 c_1 - a_3 b_1 c_2)(a_3 b_1 c_2 - a_1 b_2 c_3)(a_1 b_2 c_3 - a_2 b_3 c_1)$$
$$(a_2 b_1 c_3 - a_3 b_2 c_1)(a_3 b_2 c_1 - a_1 b_3 c_2)(a_1 b_3 c_2 - a_2 b_1 c_3).$$

The meaning of $I_6 = 0$, will be shown later on page 51.

§ 4. Reciprocal or Dual Forms.

Owing to the importance of the dual forms so derived, we shall now consider the transformation of the invariants when the role of the two triangles is interchanged; that is, when the one originally taken as the 3-line is taken as the 3-point and vice-versa. The formulae thus obtained will be found very useful, for if we know a certain projective relation exists between the 3-line and the 3-point due to the vanishing of a certain invariant form, then the vanishing of the reciprocal or dual form, (which is found immediately by applying the formulae now to be found) will indicate the existence of the same relation between the

joins of the 3-point and the meets of the 3-line. To get the dual forms in the general case would obviously be equivalent to replacing each Greek and Roman letter in each of the fundamental elements by their corresponding minors in the determinants we have named Δ_1 and D_1 respectively. Evidently in determining the dual forms we take the simplified forms of the original system, for in that case we find the algebra is long. We indicate the dual forms by primes.

$$D_1' = \begin{vmatrix} b_2c_3 - b_3c_2 & a_3c_2 - a_2c_3 & a_2b_3 - a_3b_2 \\ b_3c_1 - b_1c_3 & a_1c_3 - a_3c_1 & a_3b_1 - a_1b_3 \\ b_1c_2 - b_2c_1 & a_2c_1 - a_1c_2 & a_1b_2 - a_2b_1 \end{vmatrix}$$

$$= D_1^2$$

This is evidently a complete application of a well known Theorem

in determinants; — If each term in a square determinant be replaced by its minor, the resultant is the square determinant squared.

$$\therefore \quad \underline{I_1' = D_1^2}.$$

I_1' is of course the discriminant of the determinant D_1', with all six minors positive; — that is

$$I_1' = (b_2 c_3 - b_3 c_2)(a_1 c_3 - a_3 c_1)(a_1 b_2 - a_2 b_1)$$
$$+ (b_3 c_1 - b_1 c_3)(a_2 c_1 - a_1 c_2)(a_2 b_3 - a_3 b_2)$$
$$+ (b_1 c_2 - b_2 c_1)(a_3 c_2 - a_2 c_3)(a_3 b_1 - a_1 b_3)$$
$$+ (b_2 c_3 - b_3 c_2)(a_2 c_1 - a_1 c_2)(a_3 b_1 - a_1 b_3)$$
$$+ (b_3 c_1 - b_1 c_3)(a_3 c_2 - a_2 c_3)(a_1 b_2 - a_2 b_1)$$
$$+ (b_1 c_2 - b_2 c_1)(a_1 c_3 - a_3 c_1)(a_2 b_3 - a_3 b_2),$$
$$= \sum^3 a_1^2 b_2^2 c_3^2 - \sum^3 a_1^2 b_3^2 c_2^2$$
$$- 4\left[\sum^3 a_1 a_2 b_2 b_3 c_3 c_1 - \sum^3 a_1 a_2 b_3 b_1 c_2 c_3\right]^*.$$

the terms of the types $a_1 a_2 b_1 b_2 c_3^2$ going out.

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* In all places a figure over a summation mark, indicates the number of terms in Σ.

This can be expressed rather easily in terms of our invariants, for it is of the second degree and by observation we see the D's must enter to an odd, i.e. the first degree. Now

$$I_1 D_1 = \sum^3 a_1^2 b_2^2 c_3^2 - \sum^3 a_1^2 b_3^2 c_2^2 + 2\sum^3 a_1 a_2 b_2 b_3 c_3 c_1 - 2\sum^3 a_1 a_2 b_3 b_1 c_2 c_3 .$$

$$D_2 = \sum^3 a_1 a_2 b_2 b_3 c_3 c_1 - \sum^3 a_1 a_2 b_3 b_1 c_2 c_3$$

$$\therefore \quad \underline{I_1' = I_1 D_1 - 6 D_2} .$$

I_2' and D_2' will be developed simultaneously from the determinant :-

$$(\mathrm{I}) \begin{vmatrix} (b_3c_1 - b_1c_3)(b_1c_2 - b_2c_1) & (c_3a_1 - c_1a_3)(c_1a_2 - c_2a_1) & (a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1) \\ (b_1c_2 - b_2c_1)(b_2c_3 - b_3c_2) & (c_1a_2 - c_2a_1)(c_2a_3 - c_3a_2) & (a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2) \\ (b_2c_3 - b_3c_2)(b_3c_1 - b_1c_3) & (c_2a_3 - c_3a_2)(c_3a_1 - c_1a_3) & (a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3) \end{vmatrix}$$

The expansion of this determinant is D_2', and taken with all six signs positive it is I_2'. The problem now is to express these two forms in terms of

[illegible]

Before completing the determination of dual forms, it will be of advantage to write down a series of summations of terms of the same type of the 4th degree in the coefficients of the 3 forms, and designate them by arbitrarily chosen letters. As these summations are used frequently, the notation will be adhered to throughout the work.

Tables of Summations

$$\sum^{3} a_1^4 b_2^4 c_3^4 - \sum^{3} a_1^4 b_3^4 c_2^4 = A' - A''$$

$$\sum^{9} a_1^4 b_2^3 b_3 c_2 c_3^3 - \sum^{9} a_1^4 b_2 b_3^3 c_2^3 c_3 = B' - B''$$

$$\sum^{9} a_1^3 a_2 b_1 b_2^2 b_3 c_2 c_3^3 - \sum^{9} a_1^3 a_2 b_2 b_3^3 c_1 c_2^2 c_3 = C' - C''$$

$$\sum^{18} a_1^3 a_2 b_1 b_2 b_3^2 c_2^2 c_3^2 - \sum^{18} a_1^3 a_2 b_2^2 b_3^2 c_1 c_2 c_3^2 = D' - D''$$

$$\sum^{3} a_1^2 a_2^2 b_2^2 b_3^2 c_3^2 c_1^2 - \sum^{3} a_1^2 a_2^2 b_3^2 b_1^2 c_2^2 c_3^2 = E' - E''$$

$$\sum^{3} a_1^2 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2 - \sum^{3} a_1^2 a_2 a_3 b_1 b_2 b_3^2 c_1 c_2^2 c_3 = F' - F''$$

$$\sum^{6} a_1^3 a_2 b_2^3 b_3 c_3^3 c_1 - \sum^{6} a_1^3 a_3 b_3^3 b_2 c_2^3 c_1 = G' - G''$$

$$\sum^{9} a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 = H$$

$$\sum^{3} a_1^4 b_2^2 b_3^2 c_2^2 c_3^2 = K.$$

Of course the above terms connected with the minus sign will often offset terms connected with the plus sign. In that case we shall still further abbreviate the algebra by using

$$\sum^{6} a_1^4 b_2^4 c_3^4 = A' + A'' = A,$$

$$\sum^{18} a_1^4 b_2^3 b_3 c_2 c_3^3 = B' + B'' = \bar{B}, \text{ etc., etc.}$$

It is well to remember that if one term of any of these summations appears, all must be present. So we need write in the summations only enough terms of each type to determine coefficients. With this notation the expansion of determinant (I), and consequently the expressions for D_2' and D_2'' present no difficulty, except that of doing all [illegible]. It is unnecessary to give the expanded terms of each term in the determinant, but merely the [illegible]

the three positive terms, which we will call ϕ, and of the three negative terms which we will call θ.
[illegible]

$$\phi = E' + 3E'' - 2D' + 2C' + C'' - 5F' + 6F'' - G'$$

$$\theta = 3E' + E'' - 2D'' + C' + 2C'' + 6F' - 5F'' - G''$$

Hence adding and subtracting in turn

$$I_2' = 4E - 2D + 3C + F - G$$

$$D_2' = -2(E'-E'') - 2(D'-D'') + (C'-C'') - 11(F'-F'') - (G'-G'')$$

Now as hinted at before, we can always tell if D_1 and D_2 (or both) enter to an odd or even degree. For if all terms of the same type are positive or all negative, then the D_1's enter into every term of the invariant [illegible] even degree. If however the terms of a given type are half positive and half negative then [illegible] an odd degree. This theorem requires no proof, being evident. Hence since

D_2' is of the 4th degree in the coefficients, and contains by above theorem the D_i's to an odd degree it can contain only the terms $D_1^2 D_2$, $D_1 I_3$, $D_1 I_1 I_2$, $D_1 I_2$ and $I_1^2 D_2$.

Expressing these in terms of the summations just given we have

$D_1^2 D_2 = 2(E' - E'') + 2(D' - D'') - (C' - C'') + 11(F' - F'') + (G' - G'')$.

It is unnecessary to write down the [illegible], as we [illegible] precisely the negative of D_2'.

$\therefore \quad \underline{D_2' = -D_1^2 D_2}$.

Similarly I_2' can contain only the terms $D_1^2 I_2$, $I_1 D_1 D_2$, D_2^2 and D_1^4.* [illegible] but only the [illegible] are needed,

$D_1^2 I_2 = C - 2D + 2E - F + G + 4H$

$I_1 D_1 D_2 = -C \quad + 2E + 5F + G - 4H$

$D_2^2 = \quad E + 2F \quad - 2H$

$\therefore \underline{I_2' = D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2}$, [illegible] by a comparison with the [illegible]

* 2nd 4th degree terms [illegible] D_1^4 [illegible]

the preceding page.

Some other 4th degree forms which were of frequent use in later computations will be placed here for reference.

$$I_1^2 I_2 = 2E + 2L + C + 11F + G - 4H$$

$$I_2^2 = [illegible] - 2F - 2H$$

$$I_1 I_3 = F$$

$$I_1^4 = A + 6E + 12C + 4G + 36F + 4B + 12D + 24H + 6K$$

$$D_1^4 = A + 6E + 12C + 4G - 12F - 4B - 12D + 24H + 6K$$

$$I_1^2 D_1^2 = A + 6E + 12C + 4G + 12F + 24H - 2K.$$

The derivation of these formulae is tedious and somewhat lengthy but presents no difficulty in itself. They can be verified however by assigning to the variables simple numerical values.

Th[illegible]

[illegible] of I_3. [illegible]

$$I_3' = (b_2c_3 - b_3c_2)(b_3c_1 - b_1c_3)(b_1c_2 - b_2c_1)(c_2a_3 - c_3a_2)$$
$$(c_3a_1 - c_1a_3)(c_1a_2 - c_2a_1)(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1).$$

By merely [illegible] the [illegible] [illegible] the previous case the algebra becomes almost unmanageable. The following method, which involves the notation of the previous paragraph, seems to [illegible] [illegible] the [illegible] difficulties. The [illegible] [illegible] [illegible]

[illegible]

[illegible]

[illegible]

$\alpha_0 + \alpha_1 + \alpha_2 =$ [illegible], [illegible],

[illegible] $\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_0 = r_2$, $\beta_0\beta_1 + \beta_1\beta_2 + \beta_2\beta_0 = s_2$,

$$\alpha_0\alpha_1\alpha_2 = \beta_0\beta_1\beta_2 = r_3 = s_3 = I_3,$$

[illegible] the [illegible] [illegible]

[illegible]

[illegible]

Next group the terms of I_3' thus:

$$(2)\quad [(b_2c_3-b_3c_2)(c_3a_1-c_1a_3)(a_1b_2-a_2b_1)]$$
$$[(b_3c_1-b_1c_3)(c_1a_2-c_2a_1)(a_2b_3-a_3b_2)]$$
$$[(b_1c_2-b_2c_1)(c_2a_3-c_3a_2)(a_3b_1-a_1b_3)]$$

It is then very simple to express each of these three factors by means of the symbols first defined in

$$(3)\quad I_3' = (\alpha_0^2-\alpha_0 s_1+s_2-\alpha_1\alpha_2)(\alpha_1^2-\alpha_1 s_1+s_2-\alpha_0\alpha_2)(\alpha_2^2-\alpha_2 s_1+s_2-\alpha_0\alpha_1).$$

Expanding the right hand side, collecting and simplifying as much as possible in same symbols,

$$(4)\quad I_3' = I_3(\alpha^3) - \sum^3 \alpha_i^3\alpha_k^3 - I_3 r_2 s_1 - I_3 s_1(\alpha^2) + \sum^6 \alpha_i^2\alpha_k^3 s_1 + I_3 r_1 s_1^2 - \sum^3 \alpha_i^2\alpha_k^2 s_1^2 - I_3 s_1^3 + \sum^3 \alpha_i^2\alpha_k^2 s_2 + I_3 r_1 s_2 - \sum^6 \alpha_i\alpha_k^3 s_2 + r_2 s_1^2 s_2 - r_1 s_2^2 s_1 + s_2^3 - r_2 s_2^2 + (\alpha^2) s_2^2.$$

where $(\alpha^3) = \alpha_0^3+\alpha_1^3+\alpha_2^3$, etc

and $i = 0, 1, 2$; $k = 0, 1, 2$, $i \neq k$.

In order to get the expression entirely in terms of r_i and s_i it was found best to employ the following

identities, which can be derived by simple calculations;

$$\sum^{6} \alpha_i \alpha_k^2 = r_1 r_2^2 - I_3 r_2 - 3 I_3 r_1^2 .$$

$$\sum^{6} \alpha_i \alpha_k^2 = r_1^2 r_2 - I_3 r_1 - 2 r_2^2 .$$

$$\sum^{3} \alpha_i^2 \alpha_k^2 = r_2^2 - 2 I_3 r_1 .$$

$$(\alpha^3) = r_1^3 - 3 r_1 r_2 - 3 I_3 .$$

$$(\alpha^2) = r_1^2 - 2 r_2 .$$

$$\sum^{3} \alpha_i^3 \alpha_k^3 = r_2^3 - 3 I_3 r_1 r_2 - 3 I_3^2 .$$

Substituting these values directly in (4) we have

$$(5) \quad I_3 r_1^3 - \cancel{3 I_3 r_1 r_2} + \cancel{3 I_3^2} - r_2^3 + \cancel{3 I_3 r_1 r_2} - \cancel{3 I_3^2}$$

$$- \cancel{I_3 r_2 s_1} - I_3 r_1^2 s_1 + \cancel{2 I_3 r_2 s_1} + r_1 r_2^2 s_1 - \cancel{I_3 r_2 s_1} -$$

$$- 2 I_3 r_1^2 s_1 + I_3 r_1 s_1^2 - r_2^2 s_1^2 + 2 I_3 r_1 s_1^2 - I_3 s_1^3$$

$$+ r_2^2 s_2 - \cancel{2 I_3 r_1 s_2} + \cancel{I_3 r_1 s_2} - r_1^2 r_2 s_2 + \cancel{I_3 r_1 s_2}$$

$$+ 2 r_2^2 s_2 + r_2 s_1^2 s_2 - r_1 s_2^2 s_1 + s_2^3 - r_2 s_2^2$$

$$+ r_1^2 s_2^2 - 2 r_2 s_2^2 .$$

Collecting terms, [illegible]

$$(6) \quad I_3{}' - I_3 (r_1^3 - s_1^3) - (r_2^3 - s_2^3) - 3 I_3 (r_1^2 s_1 - r_1 s_1^2)$$

$$+ r_1 s_1 (r_2^2 - s_2^2) - ([illegible]) + 3 [illegible] (r_2 - s_2) - r_2 s_2 ([illegible]) .$$

Before we can finally reduce to our system of covariants we need the following additional identities.

$$r_2 s_1 - r_1 s_2 = \frac{1}{2}(I_1 D_2 - I_2 D_1).$$

$$r_1 r_2 - s_1 s_2 = \frac{1}{2}(I_1 I_2 + D_1 D_2).$$

$$r_2 s_1 + r_1 s_2 = \frac{1}{2}(D_1 I_2 - I_1 D_2).$$

$$r_2 s_2 = \frac{1}{4}(I_2^2 - D_2^2).$$

$$r_1 s_1 = \frac{1}{4}(I_1^2 - D_1^2).$$

Reducing (6) by steps we have

$$I_3' = D_1 I_3 (r_1^2 + r_1 s_1 + s_1^2) - D_2 (r_2^2 + r_2 s_2 + s_2^2)$$
$$- 3 D_1 I_3 r_1 s_1 + \frac{1}{4} I_2 D_2 (I_1^2 - D_1^2)$$
$$- \frac{1}{4}(I_1 D_2 - I_2 D_1)(I_1 I_2 - D_1 D_2) - 3 D_2 r_2 s_2 - \frac{1}{4} I_1 D_1 (I_2^2 - D_2^2)$$

$$= D_1^3 I_3 - D_2^3 + \frac{1}{4}(I_1^2 I_2 D_2 - I_2 D_1^2 D_2 - I_1^2 I_2 D_2$$
$$+ I_1 I_2^2 D_1 + I_1 D_1 D_2^2 - I_2 D_1^2 D_2 - I_1 I_2^2 D_1 + I_1 D_1 D_2^2)$$

$$= D_1^3 I_3 - D_2^3 + \frac{1}{4}(2 I_1 D_1 D_2^2 - 2 I_2 D_1^2 D_2).$$

or as a final form

(7) $\underline{I_3' = D_1^3 I_3 - D_2^3 + \frac{1}{2} D_1 D_2 (I_1 D_2 - I_2 D_1)}$.

Summarizing the results of this section we have the following

Table of Dual Forms.

$$
A.\left\{\begin{aligned}
I_1' &= I_1 D_1 - 6 D_2\\
D_1' &= D_1^2\\
I_2' &= D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2\\
D_2' &= -D_1^2 D_2\\
I_3' &= D_1^3 I_3 - D_2^3 + \tfrac{1}{2} D_1 D_2 (I_1 D_2 - I_2 D_1).
\end{aligned}\right.
$$

These forms are evidently not homogeneous, but may be made so by throwing in the proper power of Δ_1 (which equals unity in the canonical forms respectively taken) in each term. The dual tables therefore for the 3-line and 3-point both taken generally is

$$
B.\left\{\begin{aligned}
I_1' &= I_1 D_1 \Delta_1 - 6 D_2 \quad , \quad D_1' \Delta_1' = D_1^2 \Delta_1^2\\
I_2' &= I_2 D_1^2 \Delta_1^2 - 2 I_1 D_1 \Delta_1 D_2 + 6 D_2^2\\
D_2' &= -D_1^2 \Delta_1^2 D_2\\
I_3' &= I_3 D_1^3 \Delta_1^3 - D_2^3 - \tfrac{1}{2} D_1 \Delta_1 D_2 (I_1 D_2 - I_2 D_1 \Delta_1)
\end{aligned}\right.
$$

Owing to the amount of algebra involved in calculating the dual forms, it is well to apply a very convenient check to the work. It is evident that if we interchange the rôles of the triangles a second time, they will be back to their original rôles again. So analytically, if we carry out transformation (A) on the right hand sides of the equations (A) we ought to get back to our original system of invariants, (each multiplied of course by some power of D_1). We will denote the results of this second substitution by the "double prime" or seconds symbol.

We have

$$I_1'' = (I_1 D_1 - 6 D_2) D_1^2 + 6 D_1^2 D_2 = D_1^3 I_1$$

Comparing these with the normal transformations in the notation on [illegible],

$$D_1'' = D_1^4 ,$$

$$I_2'' = D_1^6 I_2 ;$$

$$D_2'' = D_1^6 D_2 ,$$

$$I_3'' = D_1^{[illegible]} I_3 .$$

While these formulae have no value in themselves, they furnish a reasonable guarantee of the accuracy of the work up to this point.

§ 5. Some Invariant Relations of the 3-Point and the 3-Line.

Before working out any problems it should be noted, as pointed out before, that when the 3-line is taken as the reference triangle, the Δ_1 does not appear explicitly in the invariant forms, having assumed the value unity. It is necessary, therefore, if the result is to be accurate for the general case, that the factor Δ_1 be incorporated in each term of the result to such a power as to make it of the proper degree in the Greek letters. If Δ_1 or D_1 is a factor of any resulting invariant form, it signifies at once that the condition under discussion exists when the

It means the [illegible] is simply [illegible] ively [illegible] ... [illegible] concurrent or collinear respectively. In de- termining these functions [illegible] tion etc. the factor Δ_1 usually plays no part, [illegible] assume the 3 lines [illegible] [illegible] tangent [illegible], thus tacitly assuming that the 3 lines are not concurrent, and Δ_1 is therefore unity. A few of the simpler invariant relations will be calculated directly.

1). If the 3-points are collinear,

$$D_1 = 0.$$

2) If the 3 lines are concurrent,

$$\Delta_1 = 0.$$

These two [illegible] [illegible] proof, [illegible] [illegible] of these [illegible] [illegible] at once.

3) If the 3-point is apolar to the 3-line,

$$I_1 = 0.$$

For let the 3-line be $x_1 x_2 x_3 = 0$, and the 3-point, $(a\xi)(b\xi)(c\xi) = 0$. As is customary in polar calculations regard the x's as operators, and operate with $x_1 x_2 x_3$ on $(a\xi)(b\xi)(c\xi)$ in the usual way, and ask that the [illegible] [illegible] of the 3-point. That is, the required condition is that

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 + a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1 = 0.$$

[illegible]

$$I_1 = 0.$$

4) If the 3 points and the 3 meets of the 3-line lie on a conic

$$\Delta_1^2 D_2 = 0.$$

For, taking as before the 3-line as the triangle of reference, the general conic on the 3 meets is

$$\alpha_1 x_2 x_3 + \alpha_2 x_1 x_3 + \alpha_3 x_1 x_2 = 0 .$$

If now the 3 points $(a\xi)=0$, $(b\xi)=0$ and $(c\xi)=0$ are to be on this conic the following three equations must hold simultaneously:-

$$\alpha_1 a_2 a_3 + \alpha_2 a_1 a_3 + \alpha_3 a_1 a_2 = 0$$

$$\alpha_1 b_2 b_3 + \alpha_2 b_3 b_1 + \alpha_3 b_1 b_2 = 0$$

$$\alpha_1 c_2 c_3 + \alpha_2 c_3 c_1 + \alpha_3 c_1 c_2 = 0$$

Therefore eliminating $\alpha_1, \alpha_2, \alpha_3$ from the 3 equations, the required condition is

$$\begin{vmatrix} a_2 a_3 & a_3 a_1 & a_1 a_2 \\ b_2 b_3 & b_3 b_1 & b_1 b_2 \\ c_2 c_3 & c_3 c_1 & c_1 c_2 \end{vmatrix} = 0$$

But this is nothing more than

$$D_2 = 0.$$

A study of the degree in the Greek letters in the general case show that Δ_1^2 must be annexed as a factor. Therefore in general the condition is

$$\Delta_1^2 D_2 = 0,$$

as we started to prove.

6) If the 3 lines and the 3 joins of the 3-point are on a conic

$$D_1^2 D_2 = 0.$$

For if the 6 lines are on a conic there must be a line η, such that the polo conic of η as to the 3-point touches the 3 reference lines.* For this polo conic will touch the joins of the 3-point, for the Hessian of a triangle is the triangle

*Durège – [illegible] dritter Ordnung, S. [illegible]

itself, and the polecone of any line as to a curve of the third order, touches the curve in three points. Here the characteristics of the theorem enter again naturally, lines such as touched there. It also follows directly from the first of the said reciprocal theorems that the polar conic of any point as to a cubic passes through the double points, and for the same reason the polecone of any line as to a cubic touches the double lines. Hence the polecone of η as to the 3-point curve must contain no squared terms. That is, I must find the policonic of η as to the 3-point cubic and see that the coefficients of ξ_2^2 be zero simultaneously.

Take the 3-point equation, and when expanding, it can be written as follows: —

(1) $\sum^{3} a_i b_i c_i \xi_i^3 + \sum^{6} (a_i b_i c_k + a_i b_k c_i + a_k b_i c_i)\xi_i^2 \xi_k + \sum^{6} a_i b_k c_l \xi_1 \xi_2 \xi_3$

where $i \neq k \neq l$; $i = 1, 2, 3$; $k = 1, 2, 3$; $l = 1, 2, 3$.

Take the poloconic of η as to (1) and equate the coefficients of ξ_1^3, ξ_2^3 and ξ_3^3 respectively to zero. Without writing down the complete equation of the poloconic these coefficients are respectively

$$3a_1b_1c_1\eta_1 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)\eta_2 + (a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)\eta_3 = 0$$

$$(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2)\eta_1 + 3a_2b_2c_2\eta_2 + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)\eta_3 = 0$$

$$(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)\eta_1 + (a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)\eta_2 + a_3b_3c_3\eta_3 = 0$$

In order that these three equations hold simultaneously, the eliminant obtained by eliminating η_1, η_2, η_3 must vanish. That is

$$\begin{vmatrix} 3a_1b_1c_1, & a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1, & a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1 \\ a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2, & 3a_2b_2c_2, & a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2 \\ a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3, & a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3, & 3a_3b_3c_3 \end{vmatrix} = 0$$

The expansion of this determinant is somewhat long, but many terms cancel out and the condition expressed in

This determinant vanishing reduces at once to

$$D_1^2 D_2 = 0 .$$

Since this problem is exactly the dual of 5), the same result would have been obtained by merely taking the dual of D_2 from the formula page 38. This example verifies the dualistic formulae and also the great advantage of having them.

7) The problem of Pasch triangles in a normal collineation furnishes a nice example of calculating directly a well known invariant relation in terms of our fundamental system. The theorem to be proven is:

If the two triangles are such that the 3 points in a Pasch triangle are a normal collineation having the 3 lines for a triad

Triangle

$I_1 (I_1 a_1 - [illegible] a_2) = 0.$

The proof is simple. The fixed triangle of the collineation is the triangle of reference,— the 3-line. Using a, b, c as the points to agree with our [illegible] adopted notation, the general collineation [illegible]

[illegible]

Then [illegible] the transform of a [illegible] to be on the side b c, k b is to be on the side ac and kc is on the side ab. This gives [illegible] 3 equations like

$$[illegible] \begin{vmatrix} k_1 a_1 & k_2 a_2 & [illegible] \\ b_1 & b_2 & b_3 \\ [illegible] & [illegible] & [illegible] \end{vmatrix} = 0$$

If on the expansion of these determinants we use [illegible] to denote [illegible] $|a_1 b_2 c_3|$

we can write the three equations

$$(2)\quad \begin{cases} k_1 a_1 \alpha_1 + k_2 a_2 \alpha_2 + k_3 a_3 \alpha_3 = 0, \\ k_1 b_1 \beta_1 + k_2 b_2 \beta_2 + k_3 b_3 \beta_3 = 0, \\ k_1 c_1 \gamma_1 + k_2 c_2 \gamma_2 + k_3 c_3 \gamma_3 = 0. \end{cases}$$

Now these three equations must hold for all values of k, hence eliminating the k's we have as the necessary and sufficient condition that

$$(3)\quad \begin{vmatrix} a_1\alpha_1 & a_2\alpha_2 & a_3\alpha_3 \\ b_1\beta_1 & b_2\beta_2 & b_3\beta_3 \\ c_1\gamma_1 & c_2\gamma_2 & c_3\gamma_3 \end{vmatrix} = 0$$

Expanding this determinant, replacing the Greek letters by their equivalents in Roman letters, and collecting terms, the result can be written

$$(4)\quad \sum^{6} a_1^3 b_2^3 c_3^3 - \sum^{18} [illegible] + \sum^{18} [illegible]$$

$$- 6\, a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3 = 0$$

In this the D_i's must enter to an even degree, and it turns out that

$I_1 D_1^2$ and $D_1 D_2$ are the only combinations that enter. So (4) reduces at once to $I_1 D_1^2 - 3 D_1 D_2$, or as before the condition is

$$D_1 (I_1 D_1 - 3 D_2) = 0 .$$

γ) As a final example of this section we will look for the meaning of $I_6 = 0$, or what is the same thing, prove the theorem:

<u>If the two triangles have one perspective direction</u>

$$I_6 = 0$$

As we saw I_6 is composed of six binomial factors linear in the coefficients, each factor corresponding to a particular ordering of the perspectivity. For example, by way of proof, suppose we ask that the lines joining [illegible] and [illegible] respectively

meet in a point. The equations of the three joining lines are

$$a_2 x_3 - a_3 x_2 = 0,$$

$$b_3 x_1 - b_1 x_3 = 0,$$

$$c_1 x_2 - c_2 x_1 = 0,$$

and consequently the condition that they meet in a point is

$$\begin{vmatrix} 0 & a_3 & a_2 \\ b_3 & 0 & b_1 \\ c_2 & c_1 & 0 \end{vmatrix} = 0.$$

or $(a_2 b_3 c_1 - a_3 b_1 c_2) = 0$.

And in the same way each ordering requires one factor of I_6 to vanish.

The theorem may also be stated: If the two triangles are such that there exists a conic, such that the 3 lines are the polars of the 3 points as to that conic,

$$I_6 = 0.$$

As the simple relations discussed

by Dr. Thiem could be taken up these independently and the conditions expressed in terms of our present system. But since the present system has been made complete Dr. Thiem's measurements must be directly expressible in terms of the present one. By constructing such a transformation table it becomes possible to take advantage of all work done by the former writer. In addition the comparison thus afforded is not without interest, so a paragraph will be devoted to the formation of the equivalents of these measurements in terms of the present one.

§ 6. Expression of Hesse's invariants in terms of the second system.

Hence these fundamental invariants, as derived from the canonic form and bar notation

$$I_1 = \Sigma B_{11}$$

$$I_2 = \Sigma (B_{22} B_{33} - B_{23} B_{32})$$

$$I_3 = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix}$$

The B's are in turn defined in the general case as follows

$$B_{11} = \alpha_1 (\beta\gamma)_1 + \beta_1 (\gamma\alpha)_1 + \gamma_1 (\alpha\beta)_1$$

$$B_{22} = \alpha_2 (\beta\gamma)_2 + \beta_2 (\gamma\alpha)_2 + \gamma_2 (\alpha\beta)_2$$

$$B_{22} = \alpha_1 (\beta\gamma)_2 + \beta_1 (\gamma\alpha)_2 + \gamma_1 (\alpha\beta)_2$$

$$B_{21} = \alpha_2 (\beta\gamma)_1 + \beta_2 (\gamma\alpha)_1 + \gamma_2 (\alpha\beta)_1$$

and so on for B_{33}, B_{31}, B_{13}, B_{23} & B_{32}.

The symbols $(\beta\gamma)_i$, etc., [illegible]

[illegible] expressed [illegible],

$(\beta\gamma)_i = a_i(\ldots/\beta\gamma) + b_i(\gamma\alpha/\beta\gamma) + c_i(\alpha\beta/\beta\gamma)$,
and finally

$(bc/\beta\gamma) = (b\beta)(c\gamma) + (b\gamma)(c\beta)$, where

$(b\beta)$ etc., are [illegible] [illegible] [illegible].
These two examples explained are [illegible]
to enable the reader to write out
any others [illegible] [illegible] and verify the
[illegible] [illegible] [illegible]. These formulas
simplify very materially if we
now take these [illegible] as [illegible]
examples, as follows,

$\alpha_i\beta_i\gamma_i = 0$, $\alpha_i\beta_i\gamma_k = 0 = \alpha_i\beta_k\gamma_i = \alpha_k\beta_i\gamma_i$.

$\Sigma\alpha_i\beta_k\gamma_l = 1$; $i = 1,2,3$. $k = 1,2,3$, $l = 1,2,3$; $i \neq k \neq l$.

So we need only retain terms containing $\alpha_1\beta_2\gamma_3$ (owing to symmetry) and replace it everywhere by unity. Thus we have the [illegible] results as follows,

$B_{11} = B_{22} = B_{33} = \Sigma a_i b_k c_l$,

$B_{21} = [illegible]$,

$B_{12} = [illegible]$,

[illegible].

$B_{ij} = 2(a_j b_j c_k + a_j b_k c_j + a_k b_j c_j)$.

Or we can write down, where the new invariants are those on the left and will be marked with a bar merely to avoid confusion:

$$\bar{I}_1 = -\sum^{3} B_{ii} = -3\sum^{6} a_i b_k c_l = -3 I_1;$$

$$\bar{I}_2 = \sum^{3}(B_{22}B_{33} - B_{23}B_{32})$$
$$= 3\left[\sum^{6} a_i b_k c_l\right]^2 - 4\Big[(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2) + (a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)$$
$$+ (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)\Big]$$
$$= 3I_1^2 - 4\left(\sum^{9} a_1^2 b_2 b_3 c_2 c_3 + 3\sum^{6} a_2 a_3 b_3 b_1 c_1 c_2\right)$$
$$= 3I_1^2 - (I_1^2 - D_1^2 + 12 I_2) = 2I_1^2 + D_1^2 - 12 I_2,$$

as is easily verified.

$-\bar{I}_3 = -\Delta$ [illegible] to transform and separate, we can [illegible].

$$-\bar{I}_3 = \begin{vmatrix} I_1 & 2(a_2b_2c_1 + a_2b_3c_2 + a_3b_2c_2), & 2(a_2b_3c_2 + a_3b_2c_3 + a_2b_3c_3) \\ 2(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1) & I_1 & 2(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3) \\ 2(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1) & 2(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2) & I_1 \end{vmatrix}$$

Expanding this determinant

$$= I_1^3 + 8(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2)$$
$$(a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)$$

$$+8(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)$$

$$-4I_1\big[(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)$$
$$+(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)(a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)$$
$$+(a_2b_2c_1 + a_2b_1c_2 + a_1b_2c_2)(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)\big]$$

= since the last expression was found in the preceding ~~paragraph~~, page to be equal to $\frac{1}{4}(I_1^2 - D_1^2 + 12 I_2)$

$$= I_1^3 + 16 \sum^{18} a_1a_2a_3b_2b_3^2c_1^2c_2 + 8 \sum^{12} a_2^2a_3b_3b_1^2c_1c_2^2$$
$$+ 48 a_1a_2a_3b_1b_2b_3c_1c_2c_3 - I_1(I_1^2 - D_1^2 + 12 I_2)$$

whence by direct calculation we get

$$= I_1^3 + 12 I_1 I_2 - 4 I_1 D_2 - I_1^3 + I_1 D_1^2 - 12 I_1 I_2$$

$$= I_1 D_1^2 - 4 D_1 D_2 = D_1(I_1 D_1 - 4 D_2)$$

And since $\bar{\Delta}$ and $\bar{D}$ are already the same as Δ_1 and D_1, [illegible] [illegible] [illegible]

[illegible] $\bar{I} = 1$, [illegible] I_1, and

$$-\bar{\Lambda}\,\bar{\Gamma}\,\bar{N} = \Delta_1 (I_1 \Delta_1 - [illegible])$$

$$\text{or} \quad \bar{N} = [illegible]\,\Delta_2 - I_1 \Delta_1$$

In this [illegible] we must always see that Δ_1 is inserted to a proper power in each term when the 3-line is taken generally.

Summarizing these results in the form of a table, which we can use in translating all of [illegible] in terms of the present table, we have the left [illegible]:

$$D = [illegible]_1$$

$$\Delta = [illegible]_1$$

$$\bar{I}_1 = [illegible]_1$$

$$\bar{I}_2 = [illegible]$$

$$\bar{N} = [illegible]\, I_1 \Delta_1 + [illegible]_2 .$$

[illegible] of this table we [illegible]

write down the equivalents of the results tabulated by Dr. [illegible] on pages 50 and 51 of his article.

The 3-point degenerates, if $D_1 = 0$.	The 3-line degenerates, if $\Delta_1 = 0$.
The 3 point is apolar to the 3-line, if $I_1 = 0$.	The 3-line is apolar to the 3-point, if $I_1 = 0$.
The roots of the 3-lines are apolar to the zeros of the 3 point, if $I_1 D_1 \Delta_1 - 6 D_2 = 0$.	The zeros of the 3-point are apolar to the roots of the 3-line, if $I_1 D_1 \Delta_1 - 6 D_2 = 0$.

A conic circumscribed to the 3-point and apolar to the 3-line exists if

$$I_1 D_1 \Delta_1 - 2 D_2 = 0 .$$

A conic inscribed in the 3-line and apolar to the 3-point exists if

$$I_1 D_1 \Delta_1 - 2 D_2 = 0 .$$

A point conic apolar to both the 3-point and the 3-line exists if

$$D_1^2 D_2 = 0 .$$

A line conic apolar to both the 3-line and the 3-point exists if

$$\Delta_1^2 D_2 = 0 .$$

A conic circumscribed about both the 3-point and the 3-line exists if

$$\Delta_1^2 D_2 = 0 .$$

A conic inscribed to both the 3-point and the 3-line exists if

$$D_1^2 D_2 = 0 .$$

There exists a point whose [illegible]

There exists a line [illegible]

ar conic as to the 3-line is apolar to the 3-point if $D_1 \Delta_1 (I_1 D_1 \Delta_1 - 4 D_2) = 0$.

The three polar lines as to the 3-line, of the points of the 3-point taken two at a time, meet in a point if

$$\Delta_1 (I_1 D_1 \Delta_1 - 4 D_2) = 0.$$

There exists a collineation having the 3-point as a fixed triangle, which sends each point of the 3-line into a point on the ob-

polar as to the 3-point is apolar to the 3-line if $D_1 \Delta_1 (I_1 D_1 \Delta_1 - 4 D_2) = 0$.

The three polar points as to the 3-point, of the lines of the 3-line taken two at a time, lie on a line if

$$D_1 (I_1 D_1 \Delta_1 - 4 D_2) = 0.$$

There exists a collineation having the 3-line as a fixed triangle which sends each line of the 3-point into a line through the

opposite meet if
$\Delta_1(I_1 D_1 \Delta_1 - 3 D_2) = 0$.

There exists a collineation leaving the meets of the 3-line as fixed points which sends each meet of the 3-line onto a point on the join of the other two if

$$I_1 D_1 \Delta_1 - 3 D_2 = 0.$$

There exists a point, such that its polar conic as to the joins of the 3-point is apolar to the meets of the 3-line

opposite point if
$D_1(I_1 D_1 \Delta_1 - 3 D_2) = 0$.

There exists a collineation leaving the joins of the 3-point as fixed lines which sends each line of the 3-point onto a line through the meet of the other two if

$$I_1 D_1 \Delta_1 - 3 D_2 = 0.$$

There exists a line such that its conic polar as to the meets of the 3-line is apolar to the joins of the 3-point

if	if
$D_1^2\Delta_1^2(I_1 D_1 \Delta_1 - 2 D_2) = 0.$	$D_1^2\Delta_1^2(I_1 D_1 \Delta_1 - 2 D_2) = 0.$

Another theorem adapted from Hun's article is:-

The[re] exists a line conic touching the sides of the 3-point and apolar to the 3-line if

$$D_1^2(I_1 D_1 \Delta_1 - 4 D_2) = 0.$$

As is seen from this summary, all of Hun's invariant forms are members of the pencil

$$I_1 D_1 \Delta_1 + \lambda D_2 = 0.$$

As will be more evident from subsequent discussion, if two of the points are taken at the circular points at infinity, and the third point is taken variable, this pencil becomes the pencil of circles of which the Feuerbach circle and the incircle[s] are members. It was

be observed that 'parallel' theorems in the two columns are not dual in the sense we used the term in deriving the dual forms of the fundamental invariants.

We might add here a pair of dual theorems:–

<u>If one of the 3-points lies on either of the 2 lines</u>	<u>If a root of the 3-line lies on a point of the 3-point</u>
<u>$I_3 = 0$</u>.	<u>$4D_1^3 I_3 - 2D_2^3$</u> $\cdot I_1 D_2$ [illegible] $- D_1 = 0$

The theorem on the left is almost obvious. For if $(a\xi)=0$ say is to be on $x_1=0$, $x_2=0$ or $x_3=0$, $a_1=0$, $a_2=0$ or $a_3=0$ respectively. In either case I_3 vanishes; so for $(b\xi)$ & $(c\xi)$. The theorem on the right is gotten by using the dual transformation formulae. The powers of D_1 have not been inserted.

It may be worthy of note that

we have found a simple meaning for the vanishing of each of the fundamental invariants, including $I_{[illegible]}$, with the exception of $I_{[illegible]}$. Several indirect interpretations are available, but probably the best geometrically is the one on page 134.

§ 7. Condition that a Conic may be Inscribed in one Triangle and Circumscribed about the Other.

To find the invariant condition on two triangles such that a conic may be drawn touching the three sides of one triangle and on the vertices of the other is of particular interest and will be discussed at length. An attempt to solve this problem was in fact the genesis of this article. It is in general of no special import whether we regard the triangle taken as the reference triangle as a 3-line or a 3-point. But as a matter of consistency we shall speak of having taken the 3-line as triangle

of reference, and shall speak

[illegible] of the three as a three [illegible].

We shall first take the polar conics of a, b and c each taken as to the 3-line, – the reference triangle. They are

(1). $a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 = 0.$

(2). $b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2 = 0.$

(3). $c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2 = 0.$

Now take any line η of (1), the polar conic of a as to the reference 3 line, and the polo-conic of η will pass through a.* (Clebsch calls the poloconic the conic polar.) Hence it follows that if the three polar conics, (1), (2) and (3), [illegible] a [illegible]

* Salmon-Fiedler, "Higher Plane Curves" p 202. Clebsch, "Lecons sur la Geometrie, vol II part [illegible]. See also [illegible] page 276, ff.

line, the poloconic of these lines as to the ∞-line would be on all three points a, b, and c. But further this poloconic would also touch the three reference lines, α, β, γ. For the poloconic of any line as to a curve of the third order touches the Hessian of the curve in three points.* But the Hessian, in case the curve is a triangle, is the triangle itself, and since all three sides of the triangle enter symmetrically, the poloconic would touch each line of the Hessian once. That this poloconic touches the three lines can also be shown directly in the light of the well known theorem that the polar conic of any point as to a [illegible]

* Durège. "Ebene Curven dritter Ordnung" p. 11.

the double points of the cubic. Pole-conic and polar conic are dual terms

Consequently our problem is now reduced to finding the condition that the above three polar conics may have a common line. Salmon* gives the condition that three point conics have a common point as $T^2 = 64M$, where T and M are invariants of the net of conics determined by the three base conics. Their meaning need not be repeated here. So for three line conics to have a common line we need only form the line equations of (1), (2), (3) and form the invariants for them, corresponding to T and M and express the results in our present symbols.

* Conic Sections §§ 385-6.

The line equations of the three polar conics (1), (2), (3), can be written down in the usual way as

(4). $$a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2 - 2a_2a_3\xi_2\xi_3 - 2a_3a_1\xi_3\xi_1 - 2a_1a_2\xi_1\xi_2 = 0 = U.$$

(5). $$b_1^2\xi_1^2 + b_2^2\xi_2^2 + b_3^2\xi_3^2 - 2b_2b_3\xi_2\xi_3 - 2b_3b_1\xi_3\xi_1 - 2b_1b_2\xi_1\xi_2 = 0 = V.$$

(6). $$c_1^2\xi_1^2 + c_2^2\xi_2^2 + c_3^2\xi_3^2 - 2c_2c_3\xi_2\xi_3 - 2c_3c_1\xi_3\xi_1 - 2c_1c_2\xi_1\xi_2 = 0 = W.$$

Now as in a net of point conics $M = 0$, requires that one of the net be a double line, so in a web of line conics as determined by U, V and W, $M = 0$ requires that one of the web $lU + mV + nW$ be a double point. In that case the reciprocal of that particular one of the web must vanish identically.*
[illegible] this [illegible] between [illegible]

* Salmon Conic Sections - p. 566.

for point conics, the reciprocals or duals, of

(7) $\quad l\,U + m\,V + n\,W = 0$

can be written in abridged form

(8) $l^2\Sigma + m^2\Sigma' + n^2\Sigma'' + mn\,\phi_{23} + nl\,\phi_{31} + lm\,\phi_{12} = 0$

where the Σ's are the reciprocals of U, V and W respectively and the ϕ's are the Clebschians or intermediates of the same taken two at a time. By the term Clebschian or intermediate is meant the form occurring in this way. If, in symbolic notation, $(a\xi)^2 \equiv (b\xi)^2 = 0$ is a line conic then $|abx|^2 = 0$, is the point equation,** where the symbol $|abx|$ means the three row determinant. But if $(a\xi)^2$ and $(b\xi)^2$ are distinct conics, then $|abx|^2$ is a covariant (really contravariant) point conic

** Clebsch-"Leçons" Vol I p. 345, [illegible].

which we will call the Clebschiant.*
In Salmon's non-symbolic notation** the Clebschiant of two point conics given in his well known general form is

$$\phi = (bc'+b'c-2ff')\xi^2 + (ca'+c'a-2gg')\eta^2 + (ab'+a'b-2hh')\zeta^2 + 2(gh'+g'h-af'-a'f)\eta\zeta + 2(hf'+h'f-bg'-b'g)\zeta\xi + 2(fg'+f'g-ch'-c'h)\xi\eta.$$

This form can easily be identified with the much simpler looking symbolic form. Either might be used in continuing our work. But in the present case since U, V & W are precisely $(a\xi)^2$, $(b\xi)^2$ and $(c\xi)^2$, save as to the sign of the product terms, it is obviously of advantage to use the symbolic

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* Hun. uses the term intermediate, p. 44.
See also Coble, Trans. Am. Soc. Vol IV, p. 10.

** Conic Sections, p. 344.

formulae for the Clebschian can be written down at once by expanding $|a\,b\,x|^2$, $|b\,c\,x|^2$ and $|c\,a\,x|^2$, if we make the proper changes of sign after expansion.

For Σ we would naturally expect (since it was the counterpart of (1)) that we would get

$$a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 = 0 \text{ again,}$$

but the ever-present outside factor may not be omitted in this case for reasons which are evident. So we have finally as values for the quantities in (8) the following:

(9). $\Sigma = 4 a_1 a_2 a_3 (a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2)$.

(10). $\Sigma' = 4 b_1 b_2 b_3 (b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2)$.

(11). $\Sigma'' = 4 c_1 c_2 c_3 (c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2)$.

$$(12).\quad \phi_{12} = (a_2b_3 - a_3b_2)^2 x_1^2 + (a_3b_1 - a_1b_3)^2 x_2^2 + (a_1b_2 - a_2b_1)^2 x_3^2$$
$$+ (a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)\, x_2 x_3$$
$$+ (a_1b_2 - a_2b_1)(a_2b_3 - a_3b_2)\, x_3 x_1$$
$$+ (a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)\, x_1 x_2 = 0$$

$$(13).\quad \phi_{23} = (b_2c_3 - b_3c_2)^2 x_1^2 + (b_3c_1 - b_1c_3)^2 x_2^2 + (b_1c_2 - b_2c_1)^2 x_3^2$$
$$+ (b_3c_1 - b_1c_3)(b_1c_2 - b_2c_1)\, x_2 x_3$$
$$+ (b_1c_2 - b_2c_1)(b_2c_3 - b_3c_2)\, x_3 x_1$$
$$+ (b_2c_3 - b_3c_2)(b_3c_1 - b_1c_3)\, x_1 x_2 = 0$$

$$(14).\quad \phi_{31} = (c_2a_3 - c_3a_2)^2 x_1^2 + (c_3a_1 - c_1a_3)^2 x_2^2 + (c_1a_2 - c_2a_1)^2 x_3^2$$
$$+ (c_3a_1 - c_1a_3)(c_1a_2 - c_2a_1)\, x_2 x_3$$
$$+ (c_1a_2 - c_2a_1)(c_2a_3 - c_3a_2)\, x_3 x_1$$
$$+ (c_2a_3 - c_3a_2)(c_3a_1 - c_1a_3)\, x_1 x_2 = 0.$$

Following Salmon's determinant form for M,*

$$(15)\quad M = \begin{vmatrix} A & B & C & F & G & H \\ A' & B' & C' & F' & G' & H' \\ A'' & B'' & C'' & F'' & G'' & H'' \\ A_{23} & B_{23} & C_{23} & F_{23} & G_{23} & H_{23} \\ A_{31} & B_{31} & C_{31} & F_{31} & G_{31} & H_{31} \\ A_{12} & B_{12} & C_{12} & F_{12} & G_{12} & H_{12} \end{vmatrix}$$

* Conic Sections p. 3[illegible]

where A, B, C, F, G, H are the co-efficients of x_1^2; x_2^2; x_3^2; $x_2 x_3$, $x_3 x_1$; $x_1 x_2$ respectively in Σ'. [illegible] for the other two cases, we have the coefficients of Σ', Σ'', ϕ_{23}, ϕ_{31} and ϕ_{12}.

Substituting in (15) the actual coefficients found in equations (11)-(14) we have

(16) $M =$

$$\begin{bmatrix} 0 & 0 & 0 & 2a_1^2a_2a_3, & 2a_1a_2^2a_3, & 2a_1a_2a_3^2 \\ 0 & 0 & 0 & 2b_1^2b_2b_3, & 2b_1b_2^2b_3, & 2b_1b_2b_3^2 \\ 0 & 0 & 0 & 2c_1^2c_2c_3, & 2c_1c_2^2c_3, & 2c_1c_2c_3^2 \\ (a_2b_3-a_3b_2)^2, & (a_3b_1-a_1b_3)^2, & (a_1b_2-a_2b_1)^2, & \sim & \sim & \sim \\ (b_2c_3-b_3c_2)^2, & (b_3c_1-b_1c_3)^2, & (b_1c_2-b_2c_1)^2, & \sim & \sim & \sim \\ (c_2a_3-c_3a_2)^2, & (c_3a_1-c_1a_3)^2, & (c_1a_2-c_2a_1)^2, & & & \end{bmatrix}$$

It is not necessary to write down explicitly the minors indicated by wave strokes, as their complementary factors are zeros. So M at once reduces to the product of two

These two determinants are

(17) $M =$

$$\begin{vmatrix} (a_2 b_3 - a_3 b_2)^2 & (a_3 b_1 - a_1 b_3)^2 & (a_1 b_2 - a_2 b_1)^2 \\ (b_2 c_3 - b_3 c_2)^2 & (b_3 c_1 - b_1 c_3)^2 & (b_1 c_2 - b_2 c_1)^2 \\ (c_2 a_3 - c_3 a_2)^2 & (c_3 a_1 - c_1 a_3)^2 & (c_1 a_2 - c_2 a_1)^2 \end{vmatrix}$$

$$\times \begin{vmatrix} 2 a_1^2 a_2 a_3 & 2 a_1 a_2^2 a_3 & 2 a_1 a_2 a_3^2 \\ 2 b_1^2 b_2 b_3 & 2 b_1 b_2^2 b_3 & 2 b_1 b_2 b_3^2 \\ 2 c_1^2 c_2 c_3 & 2 c_1 c_2^2 c_3 & 2 c_1 c_2 c_3^2 \end{vmatrix}$$

Each of these determinants is readily expressed in terms of our invariants. The latter one is equal to

(18)
$$8 a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= 8 I_3 D.$

The first determinant is clearly not so simple, but is recognized as the [illegible] of

$$\begin{vmatrix} c_1^2 & c_2^2 & c_3^2 \\ a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}$$

which is only of the second degree and is equal to

(19). $$I_1 D_1 - 2 D_2 .$$

Applying the last transformation formulae to this, we get the expression of the final term or factor of (17) to be

$$(I_1 D_1 - 6 D_2) D_1^2 + 2 D_1^2 D_2$$

(20). $$D_1^2 (I_1 D_1 - 4 D_2) .$$

Substituting these values for the determinants of (17) we get

(21). $$M = 8 I_3 D_1^3 (I_1 D_1 - 4 D_2)$$

Or if we wish to make this of the proper degree in the D's and Δ's for the general case

(22). $$M = 8 I_3 D_1^3 \Delta_1^3 (I_1 D_1 \Delta_1 - 4 D_2) .$$

To express T similarly we can start with either of two forms

(23). $T = \Theta_{123}^2 - 4(\Theta_{122}\Theta_{133} + \Theta_{211}\Theta_{233} + \Theta_{311}\Theta_{322}) + 12\,\Theta$, *

or the other form **

(24) $T = (a\,b'c'')^2 + 4(a\,b'f'')(a\,c'f'') + 4(b\,c'g'')(b\,a'g'')$

$+ 4(c\,a'h'')(c\,b'h'') + 8(a\,f'g'')(b\,f'g'') + 8(a\,f'h'')(c\,f''h'')$

$+ 8(c\,g'h'')(b\,g\,h')$ [illegible] $(a\,g'h'')(b\,c'f'')$ [illegible] $(b\,h'f'')(c\,a'g'')$

$- 8(c\,f'g'')(a\,b'h'') + 4(a\,b'c'')(f\,g'h'') - 8(f\,g'h'')^2$,

where

$(a\,b'c'') = ab'c'' + ab''c' + a'bc'' + a''bc' + a'b''c + a''b'c$.

and so for all similar forms.

In fact owing to the complicated algebra in either case it was deemed best to develop both independently, which we will do in due course. Taking (24) first we must form C^2, U and UV

$a, b, c, f, g, h = a_1^2, a_2^2, a_3^2, -a_2a_3, -a_3a_1, -a_1a_2$

$a', b', c', f', g', h' = b_1^2, b_2^2, b_3^2, -b_2b_3, -b_3b_1, -b_1b_2$

* Conic Sections, p. 331. ** Ibid, p. 335.

a', b', c', [illegible]

respectively. Making these substitutions directly in (24), and collecting the terms, which is a long task we have this expression

$$T = \sum^{6} a_1^4 b_2^4 c_3^4 + 2\sum^{4} a_1^4 b_2^2 b_3^2 c_2^2 c_3^2 + 2\sum^{6} a_1^2 a_2^2 b_2^2 b_3^2 c_3^2 c_1^2$$
$$+ 4\Big[2\sum^{4} a_1^4 b_2^2 b_3^2 c_2^2 c_3^2 + \sum^{36} a_1^2 a_2 a_3 b_1^2 b_3^2 c_2^3 c_3 - \sum^{18} a_1^4 b_2 b_3^3 c_2^3 c_3$$
$$- 2\sum^{9} a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 - 3\sum^{6} a_1^2 a_2^2 b_2^2 b_3^2 c_3^2 c_1^2\Big]$$
$$+ 8\Big[3\sum^{6} a_1^2 a_2 b_2^2 b_3^2 c_3 c_1^2 + 3\sum^{6} a_1 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2 + \sum [\text{illegible}]\, a_2^3 a_3 b_1 b_2 b_3^3 c_1 c_3$$
$$- 2\sum^{9} a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 - \sum^{36} a_1 a_2 a_3 b_1^2 b_3^2 c_2^3 c_3\Big]$$
$$+ 8\Big[\sum^{36} a_1^2 a_2 a_3 b_1^2 b_3^2 c_2^3 c_3 + 3\sum^{6} a_1^2 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2$$
$$- 4\sum^{9} a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 - \sum^{18} a_1 a_2^3 b_1 b_3^3 c_1^2 c_2 c_3\Big]$$
$$+ 4\Big[\sum^{9} a_1^2 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2 + \sum^{12} a_1^3 a_3 b_1 b_2^3 c_2 c_3^3 - \sum^{18} a_2^3 a_3 b_1^3 b_2 c_1 c_2 c_3^2\Big]$$
$$- 8\Big[\sum^{4} a_1^2 a_2 b_2 b_3^2 c_3 c_1 + 2\sum^{6} a_1^2 a_2 a_3 b_1 b_2^2 b_3 c_1 c_2 c_3^2 - 2\sum^{7} a_2^2 a_3^2 b_1 b_2 b_3 c_1^2 c_2 c_3\Big].$$

Collecting similar terms and using the abbreviations of § 4, p. 30.

$$T = A - 4B + 4C + 4D + 6E + 28F - 4G - 40H + 6K.$$

Recalling from same paragraph

$$D_1^4 = A - 4B + 12C - 12D + 6E - 12F + 4G + 24H + 6K.$$

$$D_1^2 I_2 = C - 2D + 2E - F + G + 4H.$$

$D_2^2 = \quad E + 2F \quad . H.$

We have Γ finally evaluated in the desired form,

$$(25) \qquad \Gamma = D_1^4 - 8D_1^2 I_2 + 16 D_2^2 .$$

The above method is connected with long and tedious algebra, but is a very desirable simplification of the form derived by starting with (23), i.e.

$$(26) \quad T = \theta_{123}^2 - 4(\theta_{122}\theta_{133} + \theta_{211}\theta_{233} + \theta_{311}\theta_{322}) + 12 \Theta$$

Recall what these θ's are as defined in Salmon. If one of a net of conics is

$$l(\alpha x)^2 + m(\beta x)^2 + n(\gamma x)^2 = 0,$$

and we write the discriminant

$$\begin{vmatrix} l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2, & l\alpha_1\alpha_2 + m\beta_1\beta_2 + n\gamma_1\gamma_2, & l\alpha_1\alpha_3 + m\beta_1\beta_3 + n\gamma_1\gamma_3 \\ l\alpha_1\alpha_2 + m\beta_1\beta_2 + n\gamma_1\gamma_2; & l\alpha_2^2 + m\beta_2^2 + n\gamma_2^2, & l\alpha_2\alpha_3 + m\beta_2\beta_3 + n\gamma_2\gamma_3 \\ l\alpha_1\alpha_3 + m\beta_1\beta_3 + n\gamma_1\gamma_3; & l\alpha_2\alpha_3 + m\beta_2\beta_3 + n\gamma_2\gamma_3, & l\alpha_3^2 + m\beta_3^2 + n\gamma_3^2 \end{vmatrix}$$

Then we can define:

θ_{123} is the coefficient of lmn in ...

these determinants.

θ_{122} is the coefficient of lm^2, etc.

It is unnecessary to write out all the θ's for the general case. We shall write out one or two to illustrate the process.

$$\theta_{123} = \sum^6 \alpha_1^2\beta_2^2\gamma_3^2 + 2\sum^6 \alpha_1\alpha_2\beta_2\beta_3\gamma_3\gamma_1 - 2\sum^9 \alpha_1^2\beta_2\beta_3\gamma_2\gamma_3$$

Now for our conics U, V and W, we evidently replace the Greek letters by the corresponding Roman ones, and observe that the product terms have different signs. That is $\alpha_1\alpha_2$ must be replaced by $-a_1a_2$, and so on.

This gives at once

$$\theta_{123} = \sum^6 a_1^2b_2^2c_3^2 - 2\sum^6 a_1a_2b_2b_3c_3c_1 - 2\sum^9 a_1^2b_2b_3c_2c_3$$

which can at once be expressed in determinant form as

[illegible] θ_{123} [illegible] D_1^2 [illegible]

In transforming θ_{122} and similar terms, care must be taken with the

[illegible] of [illegible]. For example, when $\beta_1\beta_2$ [illegible] [illegible] β_3^2, the sign would ap-parently be minus in the transformed form, i.e. $-b_1 b_2 b_3^2$, whereas a study of its composition shows that its is actually $b_1 b_2 b_3^2$. Observing this we get, writing

$$\theta_{122} = \alpha_1^2\beta_2^2\beta_3^2 + \alpha_2^2\beta_1^2\beta_3^2 + \alpha_3^2\beta_1^2\beta_2^2$$

$$+ 2(\alpha_2\alpha_1\beta_3\beta_2\beta_1\beta_3 + \alpha_3\alpha_2\beta_2\beta_1\beta_1\beta_3 + \alpha_1\alpha_3\beta_3\beta_2\beta_2\beta_1)$$

$$- ([illegible])$$

$$- 2(\alpha_1\alpha_3\beta_2^2\beta_3\beta_1 + \alpha_2\alpha_3\beta_1^2\beta_3\beta_2 + \alpha_2\alpha_1\beta_3^2\beta_1\beta_2),$$

[illegible] replacing Greek letters by the corresponding Roman ones, with sign changed as indicated, we have as a final form [illegible] terms:

$$\theta_{122} = -4(a_1 a_2 b_1 b_2 b_3^2 + a_2 a_3 b_2 b_3 b_1^2 + a_3 a_1 b_3 b_1 b_2^2).$$

Similarly

$$\theta_{133} = -4(a_1 a_2 c_1 c_2 c_3^2 + a_2 a_3 c_2 c_3 c_1^2 + a_3 a_1 c_3 c_1 c_2^2).$$

$$\theta_{211} = -4(b_1 b_2 a_1 a_2 a_3^2 + b_2 b_3 a_2 a_3 a_1^2 + b_3 b_1 a_3 a_1 a_2^2).$$

$$\theta_{233} = -4(b_1 b_2 c_1 c_2 c_3^2 + b_2 b_3 c_2 c_3 c_1^2 + b_3 b_1 c_3 c_1 c_2^2).$$

θ_{311} [illegible]

$$\theta_{322} = -4\,(c_1c_2\,b_1b_2\,b_3^2 + c_2c_3\,b_2b_3\,b_1^2 + c_3c_1\,b_3b_1\,b_2^2).$$

Forming now from these the products required in (23) we get

$$\theta_{122}\,\theta_{133} = 16\,\big(a_1^2a_2^2\,b_1b_2b_3^2\,c_1c_2c_3^2 + a_2^2a_3^2\,b_1^2b_2b_3\,c_1^2c_2c_3 + a_3 \text{[illegible]}$$
$$+ a_1a_2^2a_3\,b_1^2b_2b_3\,c_1c_2c_3^2 + a_1^2a_2a_3\,b_1b_2^2b_3\,c_1c_2c_3^2 + a_1a_2a_3^2\,b_1^2b_2b_3\,c_1c_2^2c_3$$
$$+ a_1a_2^2a_3 \text{[illegible]} + \text{[illegible]}\big).$$

$$\theta_{211}\,\theta_{233} = 16\,\big(a_1a_2a_3^2\,b_1^2b_2^2\,c_1c_2c_3^2 + a_1^2a_2a_3 \text{[illegible]}$$
$$+ \sum^{6} a_1^2a_2a_3\,b_1b_2^2b_3 \text{[illegible]}\big).$$

$$\theta_{311}\,\theta_{322} = 16\,\big(\text{[illegible]}$$
$$+ \sum^{6} a_1^2a_2a_3\,b_1b_2^2b_3\,c_1c_2c_3^2\big).$$

Hence

$$(\theta_{122}\,\theta_{133} + \theta_{211}\,\theta_{233} + \theta_{311}\,\theta_{322}$$
$$= 16\,\Big(\sum^{9} a_1a_2^2 \text{[illegible]} + 3\sum^{6} a_1a_2a_3 \text{[illegible]}\Big),$$

(25) $$= 16\left[\tfrac{1}{4}(I_2^2 - D_2^2) + 3I_1I_3\right]$$

$$= 4I_2^2 - 4D_2^2 + 48I_1I_3.$$

As a final step [illegible] expressing T, we must find an expression for H in terms of our invariants. H or H_{123} is defined by [illegible] the [illegible] the discriminant of

$$l\Sigma + m\Sigma' + n\Sigma''.$$

Taking values of Σ, Σ' and Σ'' from (9) (10) and (11), the discriminant can be written down from Salmon's* determinant form as

$$8\, a_1 a_2 a_3\, b_1 b_2 b_3\, c_1 c_2 c_3 \times \begin{vmatrix} 0 & la_3+mb_3+nc_3 & la_2+mb_2+nc_2 \\ la_3+mb_3+nc_3 & 0 & la_1+mb_1+nc_1 \\ la_2+mb_2+nc_2 & la_1+mb_1+nc_1 & 0 \end{vmatrix}$$

$$=16\, a_1a_2a_3 b_1b_2b_3 c_1c_2c_3 (la_1+mb_1+nc_1)(la_2+mb_2+nc_2)(la_3+mb_3+nc_3).$$

In this the coefficient of lmn is found to be

$$16\, a_1a_2a_3 b_1b_2b_3 c_1c_2c_3 (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1).$$

$$= 16\, I_3 I_1 \quad . \qquad \text{So we have}$$

(26) $\quad \Theta = 16\, I_1 I_3.$

Now substituting these values (24) (25) and (26) in the expression (23) for T we have

* Conic Sections, p. 266.

$$T = (D_1^2 - 4I_2)^2 - 4(I_2^2 - D_2^2 + 4 I_1 I_3) + 192 I_1 I_3 .$$

On collecting terms

(27) $T = D_1^4 - 8D_1^2 I_2 + 16 D_2^2$ which is the same as the expression (25) obtained by the laborious expansion of (24).

We can now substitute the values (21) and (27) for M and T in Salmon's condition $T^2 = 64 M$ and have as a final Theorem:–

<u>If the 3-point and the 3-line are such that a conic may be inscribed in the 3-line and circumscribed about the 3-point, then</u>

(28) $(D_1^4 - 8D_1^2 I_2 + 16 D_2^2)^2 - 512 I_3 D_1^3 (I_1 D_1 - 4 D_2) = 0 .$

If we now apply the dual transformation to (28) we get another theorem which leads to some interesting conclusions: The dual form is

$$\left[D_1^8 - 8D_1^4(D_1^2 I_2 - 2I_1 D_1 D_2 + 6D_2^2) + 16 D_1^4 D_2^2\right]^2$$

$$-512 D_1^6 (D_1^3 I_3 - D_2^3 + \tfrac{1}{2} I_1 D_1 D_2^2 - \tfrac{1}{2} I_2 D_1^2 D_2)(I_1 D_1^3 - 2D_1^2 D_2) = 0$$

Expanding and collecting, this becomes

$$(29).\quad D_1^{11}\Big[D_1(D_1^4 + 64 I_2^2 + 32 I_1 D_1 D_2 - 16 D_1^2 I_2 - 64 D_2^2)$$
$$- 512 I_3 (I_1 D_1 - 2 D_2)\Big] = 0.$$

Hence the dual theorem reads:-

A conic can be drawn on the meets of the 3-line and touching the joins of the 3-point if (29) = 0.

It may be noted [illegible] that no reference has been made to the introduction of Δ_1, so as to render the forms general. In fact it is scarcely necessary, for obviously in (28) there will be no extraneous factor Δ_1^i. [illegible] mechanically [illegible] vidual term the proper power of Δ_1 [illegible]

(30). $(D_1^4\Delta_1^4 - 8D_1^2\Delta_1^2 I_2 + 16D_2^2)^2 - 512\, I_3 D_1^3\Delta_1^3 (I_1 D_1\Delta_1 - 4D_2) = 0$

[illegible] vanish accordingly, hence we would have

(31) $D_1^{11}\big[D_1\Delta_1(D_1^4\Delta_1^4$ [illegible] $)$

$- 512\, I_3 (I_1 D_1 \Delta_1 - 2D_2)\big] = 0.$

It is to be noted that the essential factor of (31) is the quintic factor in the square brackets. The other factor merely expresses the fact that the construction is always possible if the three points are on a line.

We shall next consider (28) and (29) under the special condition that two of points of the 3-point are taken as I and J, the circular points at infinity, and the third point is taken as a variable point x. Then (28) and (29) become equations of the 8th and 5th degree respectively. It is this octavic and this quintic which

we shall discuss briefly. We shall designate them by R and Q respectively. That is

$$R = (D_1^4 - 8D_1^2 I_2 + 16 D_2^2)^2 - 512 I_3 D_1^2 (I_1 D_1 - 4 D_2) = 0$$

$$Q = D_1(D_1^4 + 64 I_2^2 + 32 I_1 D_1 D_2 - 16 D_1^2 I_2 - 64 D_2^2) - 512 I_3 (I_1 D_1 - 2 D_2) = 0$$

where we understand b_i and c_i are replaced by the coordinates of I and J, and a_i by x_i.

In the case of conics on the 3-line and on the 3-points, if two of the points are taken at I and J, obviously all the conics become circles on the 3-line, i.e. touching the three lines. In other words $R = 0$, is merely the locus of points, from which circles can be drawn touching the three lines of the 3-lines. This in turn says that $R = 0$ is simply the equation of the four circles which

touch the three lines and are shown in the adjoining figure being very easily constructed.

If we look at Q, we see that we have a quintic equation. In this case the conics touch~~es~~ the line joining I and J, - the line at infinity. Hence they are parabolas. Further the lines joining I and J to x are tangents to the corresponding parabola. Hence x being the meet of the tangents from I and J is the focus. So we have the theorem:-

<u>The locus of the foci of all parabolas on three points is a quintic curve, and if the three</u>

point was taken as the vertex of the reference triangle the equation of this quartic locus is

$$Q = 0.$$

It is possible to read off some facts in regard to R and Q without working out the explicit equations, although the above method gives a rather simple way of doing this. It is well to note first what curves are represented by certain simple covariants when as above the points are taken as I and J and the third varies.

$I_3 = 0$, is equation of reference triangle.

$D_1 = 0$, is equation of the line at infinity.

$I_1 = 0$, is equation of the polar line of I and J as to the triangle.

$L_2 = 0$ is equation of circumcircle.

$I_1 D_1 - 2 D_2 = 0$, is equation of the apolar circle.

$I_1 D_1 - 4 D_2 = 0$ is Feuerbach circle.

Of these forms none requires any special proof except possibly the last. By Hun,* the equation of the Feuerbach conic, or in the present case the Feuerbach circle is $N = 0$. But we saw $-N = I_1 D_1 - 4 D_2$. Or it can be shown directly from the fact that $I_1 D_1 - 4 D_2$ is the locus of centers of rectangular hyperbolas on the vertices of the 3-line. But the

feet of the 3 perpendiculars are clearly centers of such hyperbolas. But they are also points of the 9-point or Feuerbach circle. Hence since $I_1 D_1 - 4 D_2$ is a circle it must be the Feuerbach circle.

* Page 77, [illegible].

Now take the equation

$$R = [D_1^4 - 8D_1^2 I_2 + 16 D_2^2]^2 - 512 I_3 D_1^3 (I_1 D_1 - 4 D_2) = 0,$$

which we know is the equation of the four touching circles.

If $I_3 = 0$, $(D_1^4 - 8D_1^2 I_2 + 16 D_2^2)^2 = 0$, hence $I_3 = 0$, The reference triangle touches the octavic, where it (the reference triangle) cuts the quartic $D_1^4 - 8D_1^2 I_2 + 16 D_2^2 = 0$.

If $D_1 = 0$, $(D_1^4 - 8D_1^2 I_2 + 16 D_2^2)^2 = 0$, which merely says that the quartic passes through the multiple point of R, where the line at infinity cuts.

If $I_1 D_1 - 4 D_2 = 0$ the same quartic squared equals zero. Hence the Feuerbach circle touches the octavic, or passes through double points, where it is cut by the quartic, $D_1^4 - 8D_1^2 I_2 + 16 D_2^2 = 0$. This is a verification of the circle circum theorem that the Feuerbach circle touches the inscribed and escribed

curves.

This quartic which turns up in this connection is pretty interesting in itself. Writing its equation

$K = I_1^4 - 8 I_1^2 I_2 + 16 I_2^2 = 0,$

we see that when $I_1 = 0$, $I_2 = 0$, which with earlier considerations, shows that K is a bicircular quartic. We know all its intersections with the octavic;— 12, where the reference triangle touches the octavic, 4, where the Feuerbach circle touches it, and 8 at each circular point at infinity. But it is not of importance and we shall not propose to discuss this quartic in detail. The conic, $D_1^2 - 8 I_2 = 0$, which is met at several places is probably also of slight geometrical interest. The [illegible] itself being completely known, we shall take up it briefly.

Taking the form

(32) $Q = D_1[(D_1^2 - 8I_2)^2 + 32 D_1 I_1 I_2 (4 D_2)] - 512 I_3 (I_1 D_1 - 2D_2) = 0$

we can abbreviate by setting

$D_1^2 - 8I_2 = U$

$16(I_1 D_1 - 2D_2) = V$, where $U = 0$, $V = 0$ are conics. We have then

(33). $Q = D_1(U^2 - D_2 V) - 32 I_3 V = 0$

If in this $D_1 = 0$, then $I_3 V = 0$, that is Q cuts the line at infinity in the five points where I_3 and V cut it. But $I_3 = 0$ is the reference triangle and $V = 0$ is the apolar circle. Hence

The quintic $Q = 0$, passes through the circular points at infinity, and has its asymptotes parallel to the sides of the reference 5-line.

Again if $V = 0$ $D_1 U^2 = 0$, which says that $V = 0$, the apolar conic touches the quintic (or meets it at in the points) where U and V inter-

-ant. Since Q must be permanent, and its six double points which are not generated by V in a con-ic, it follows that V does not cut Q in double points. Hence the theorem:

The apolar circle $V = 0$, has four-fold contact with the quartic $Q = 0$, touching it at the four points where $U = 0$ and $V = 0$.

This same fact can be proven more directly by taking the [illegible] of $V = 0$, which comes out to be

$$I_1 U_1 - 4 U_2 = 0, \text{ say } F = 0$$

and we have proven this to be the Feuerbach circle. But it is a well known theorem that the Feuerbach circle of any triangle touches each of the four tangent circles. That is $F = 0$ touches $Q = 0$ in the four points

[illegible] destroyed [illegible] Q is the dual of R, and V of F, it follows that V touches Q at four points, which was theorem to be proved.

Suppose we rewrite Q thus

$$(34) \quad Q = D_1 U^2 + 2V(D_1 D_2 - 16 I_3) = 0$$

$$\text{or} \quad = D_1 U^2 + 2VW = 0$$

$W = 0$ is a cubic meeting the line at infinity where the sides of the reference triangle meet it and passing through the vertices of the reference triangle. [illegible] we have the theorem

<u>$W = 0$ touches Q where V cuts W.</u>

Of course these contacts are real or imaginary according as the intersections of U and W are real or imaginary. This remark applies elsewhere.

We can see in this way that where $Q = 0$ cuts the circumcircle. For in Q, let $D_2 = 0$ and it becomes

(35) $D_1(D_1^2 \ldots I_2) = \ldots I_1 \neq 0$ [illegible]

so unless the 3 points are on a line

(36). $(D_1^2 - 8 I_2)^2 - 512 I_1 I_3 = 0$

Also in $R = 0$, let $D_2 = 0$ and it becomes $(D_1^4 - 8 D_1^2 I_2)^2 - 512 I_1 I_3 D_1^4 = 0$ or dropping D_1^4 as a factor,

(37) $(D_1^2 - 8 I_2)^2 - 512 I_1 I_3 = 0. \quad = (36).$

Hence the theorem, -

<u>Q and R cut the circumcircle in the same points.</u>

If the triangle is real, six of these points are always real and can be constructed very easily. We will not go into any further discussion of this or other locus problems. But there is one very im-

mediate and simple remark which may be made. It concerns the question of whether a locus, obtained by allowing one point to vary, is on the fundamental points or not. The term fundamental points here applies to both the fixed points of the 3-point and meets of the 3-lines. If a locus is on a fundamental point, it denotes that if any variable point coincides with that fundamental point, the invariant condition is satisfied. To illustrate this take the fundamental invariants.-

$D_1 = 0$, is the equation of the join of the two fixed points, hence D_1 vanishes if the variable point coincides with a fixed point but does not vanish due to a coincidence of x with a meet of the 3-lines.

Similarly.

I_1 vanishes at no time, due to a coincidence of x with a fundamental point.

D_2 vanishes if x coincides with a point of the 3-lines, and also if it coincides with a fixed point.

I_3 does not vanish due to any coincidence of x with a fundamental point.

I_4, vanishes if x is anywhere on any of the three lines, but not when it coincides with a fixed point.

These facts are of importance at times as they enable us to tell whether a certain invariant vanishes because of a particular position of one point or whether the vanishing or non-vanishing of the invariant is unaffected by such coincidence. For instance

we know that any invariant built up of forms containing a D, or a D, vanishes on each of these, if two of the points coincide; while any invariant built up of combinations of L, and L, will not vanish if two points are taken coincident, nor if one point is taken at a vertex or node of the 3 lines.

§ 8. The Clebschians and their Invariants.

In the case of two general line cubics, the Clebschian has been defined analytically thus. Let $(a\xi)^3$ and $(b\xi)^3$ be the cubics, then

(1) $$\chi = |a\,b\,x|^3$$

is the Clebschian or intermediate.

If we were now to write the point equations of ~~the same~~ two cubics, say $(\alpha x)^3$ and $(\beta x)^3$, we would in an entirely analogous manner define the line cubic

(2) $$\chi' = |\alpha\,\beta\,\xi|^3 = 0$$

as the Clebschian of the two point cubics. In general χ and χ' are not the same for two triangles.

In forming Clebschians, it is obvious that we must take both tri-

angles in points (and both in lines), in this section. Of course the line equation $x_1 x_2 x_3$, has as its equation in line coordinates $\xi_1 \xi_2 \xi_3$. Hence starting with one cubic as the reference triangle, and the other as a general cubic $(a\xi)^3 = 0$, we have by simply substituting in the expanded form of (1)

$$(3).\ \chi = 6(a_2 a_3^2 x_1^2 x_2 + a_1 a_2^2 x_1 x_3^2 + a_1^2 a_3 x_2^2 x_3 - a_1 a_3^2 x_1 x_2^2 - a_1^2 a_2 x_2 x_3^2 - a_2^2 a_3 x_1^2 x_3).$$

If now the second cubic is taken as the 3-point $(a\xi)(b\xi)(c\xi)$, we must replace

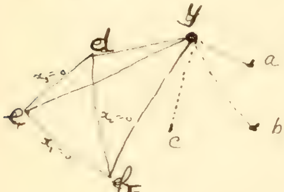
$3 a_2 a_3^2$ by $(a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2)$

$3 a_1 a_2^2$ by $(a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1)$ etc

Hence we have

$$(4)\ \chi = 2\big[(a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2) x_1^2 x_2 + (a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1) x_1 x_3^2 + (a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3) x_2^2 x_3 - (a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1) x_1 x_2^2 - (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2) x_2 x_3^2 - (a_3 b_2 c_2 + a_2 b_3 c_2 + a_2 b_2 c_3) x_1^2 x_3\big].$$

Before going further it may be desirable to fix the idea of the Clebschean of two triangles. When the triangles are both taken as 3-points, the Clebschean may



be defined as the locus of points y such that the two triads of lines y-abc and y-def are apolar. The other Clebschean is the locus of lines η having the dual property. From (4) we can read directly some of the properties of this curve. Since it does not contain any cube terms, it is on the vertices of the reference triangle, and from the reciprocality of the relations in the present case, we have the theorem—

The Clebschian of two 3-points is a cubic curve on the six points.

Again since the Clebschian has no term in $x_1 x_2 x_3$, it follows that the reference triangle is apolar to it. Hence since both triangles enter the same we have.

The Clebschian of two 3-points is a cubic curve to which both 3-points are apolar.

The dual theorem holds for the Clebschian of two 3-lines.

Going back to (4) we shall next express the invariants S and T of this cubic in terms of our fundamental system. This must clearly be possible for S and T are also invariants of the original triangles.

We shall make use of

[illegible] * [illegible] 5

$$(5') \quad S = abcm - (bc\,a_2a_3 + ca\,b_1b_3 + ab\,c_1c_2)$$
$$- m(a\,b_3c_2 + b\,c_1a_3 + c\,a_2b_1) + (a\,b_1c_2^2 + b\,c_2a_3^2$$
$$+ a\,c_1b_1^2 + b\,a_2c_1^2 + c\,b_3a_2^2 + c\,a_3b_1^2) - m^2 + 2m(b_1c_1$$
$$+ c_2a_2 + a_3b_3) - 3m(a_2b_3c_1 + a_3b_1c_2) - (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2)$$
$$+ (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2).$$

Or the corresponding coefficients [illegible]

[illegible]

$$a_2 = (a_2b_3c_3 + a_3b_2c_3 + abc).$$
$$a_3 = -(a_3b_2[illegible] + [illegible]b_1c_2 + [illegible]).$$
$$b_1 = -(a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1).$$
$$b_3 = (a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3).$$
$$c_1 = -(a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2).$$
$$c_2 = (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1).$$

It will be observed that we have dropped the common non-essential numerical factor. Making these substitutions in (5)

* Hohere Ebene Curven, [illegible]

$$(6)\quad S = -(a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1)^2 (a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1)^2$$

$$- (a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2)^2 (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2)^2$$

$$- (a_3 b_2 c_2 + a_2 b_3 c_2 + a_2 b_2 c_3)^2 (a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3)^2$$

$$+ (a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2)(a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2)$$

$$(a_3 b_2 c_2 + a_2 b_3 c_2 + a_2 b_2 c_3)(a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3)$$

$$+ (a_3 b_2 c_2 + a_2 b_3 c_2 + a_2 b_2 c_3)(a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3)$$

$$(a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1)(a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1)$$

$$+ (a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1)(a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1)$$

$$(a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2)(a_2 b_3 c_3 + a_3 b_2 c_3 + a_3 b_3 c_2).$$

The actual work of expanding and collecting the terms in (6) is long but the result can be very concisely expressed, for many terms cancel:-

$$(7)\quad S = 16\left[\sum_1^{18} a_1 a_2 \ldots b_2^2 \ldots c_2 c_3^3 - \sum^9 a_1^2 a_2^2 b_1 b_2 b_3^2 c_1 c_2 c_3^2 - \sum^9 a_1^4 b_2^2 b_3^2 c_2^2 c_3^2\right].$$

We can here again use the summation symbols of § 4 to advantage, for we have

$$(8)\quad S = 16(C - H - K).$$

We observe that the D_c's must enter to an even degree (or not at all) in every term. Combining the equalities found § 4, page 30 we find the only combinations required are

$$I_2(D_1^2+I_1^2) = 2C + 4E + 16F + 2G + 8H = P$$

$$I_2^2 - D_2^2 = 4H = P'$$

$$(I_1^2 - D_1^2)^2 = 32C + 64H + 16K = P''$$

$$I_1 D_1 D_2 = -C + 2E + 5F + G - 4H = P'''$$

It remains to properly choose coefficients, which is relatively easy, being the solution of many simple simultaneous equations in expressions of the type

(7) $\lambda P + \mu P' + \nu P'' + \rho P'''$.

It turns out that $\lambda = 12$, $\mu = -36$, $\nu = 1$ and $\rho = -24$.

We can verify this directly, substituting these values of λ, μ, ν, ρ in (7) [illegible]

[illegible]

$$12P = 24C + 48E + 120F + 24G + 96H$$

$$24P''' = 24C - 48E - 120F - 144G + 96H$$

$$-36P' = \qquad -144H$$

$$-P'' = -32C \qquad -64H - 16I$$

$$\therefore 12P - 36P' - P'' - 24P''' = 16C - 16H - 16I = S.$$

or.

(10). $S = 12 I_2 (D_1^2 + I_1^2) - 36 (I_2^2 - D_2^2) - (I_1^2 - D_1^2)^2 - 24 I_1 D_1 D_2$

where $S = 0$, is the condition that the vertices of the Clebschian lie on three lines.

Owing to the fact that the Clebschian treats the two triangles symmetrically, it is clear that S should be self-dual and a trial as to the truth of this is a good check on the work. The dual is

(11) $S' = 12(I_2 D_1^2 - 2 I_1 D_1 D_2 + 6 D_2^2)(D_1^4 + I_1^2 D_1^2 - 12 I_1 D_1 D_2 + 36 D_2^2)$

$\quad - 36[(D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2)^2 - D_1^4 D_2^2] - (I_1^2 D_1^2 - 12 I_1 D_1 D_2 + 36 D_2^2 - D_1^4)^2$

$\quad + 24(I_1 D_1^2 - 6 D_2^2) D_1^4 D_2$

(12) $= D_1^4 S.$

To express S in somewhat more concise form we can write it as the difference of two squares;

(13). $S = 4(I_1 D_1 - 3D_2)^2 - (I_1^2 + D_1^2 - 6I_2)^2.$

$I_1 D_1 - 3D_2$ is one of ~~three~~ two self dual forms, and in the special case when two points are taken at I and J and the third at o, it is the circle having the join of the ortho-center and the centroid as a diameter.

The invariant T of a cubic curve is written out in full by Salmon† for the general case. If we substitute the coefficients of χ, the Clebschian directly into the form there given, and attempt to express the result, by direct processes in terms of our invariants, the algebra becomes very cumbersome, if not entirely unmanageable.

† Salmon: Higher Plane Curves, page [illegible]

[illegible]. The following method while not very brief, presents few difficulties.

Take as the cubic [illegible] as the reference triangle, and the other as a general cubic, in Salmon's form

(14). $a\xi_1^3 + b\xi_2^3 + c\xi_3^3 + 3a_2\xi_1^2\xi_2 + 3a_3\xi_1^2\xi_3 + 3b_1\xi_2^2\xi_1$
$\quad + 3b_3\xi_2^2\xi_3 + 3c_1\xi_3^2\xi_1 + 3c_2\xi_3^2\xi_2 + 6m\xi_1\xi_2\xi_3.$

Now following Clebsch, we will write the connex set up by two general cubics $(\alpha x)^3$ and $(a\xi)^3$,

$$(\alpha a)^2(\alpha x)(a\xi) = 0$$

and follow his method of getting the invariants of the connex. The fixed points of the connex are given by the three equations

(15). $a_i(\alpha x)(\alpha a)^2 = \lambda x_i \qquad i = 1, 2, 3.$

which require in order to be consistent that $\Delta(\lambda)$ vanish, where

$$(\underline{16}) \quad \Delta(\lambda) = \begin{vmatrix} a_1\alpha_1(\alpha a)^2+\lambda, & a_2\alpha_1(\alpha a)^2, & a_3\alpha_1(\alpha a)^2 \\ a_1\alpha_2(\alpha a)^2, & (a_2\alpha_2)(\alpha a)^2+\lambda, & a_3\alpha_2(\alpha a)^2 \\ a_1\alpha_3(\alpha a)^2, & (a_2\alpha_3)(\alpha a)^2, & a_3\alpha_3(\alpha a)^2+\lambda \end{vmatrix}$$

and the invariants of the cubics are the coefficients of the powers of λ in the expansion of this determinant, and are in fact the particular invariants (in the case of two triangles) discussed by Dr. Hun.

As soon as one of the cubics is taken as the reference 3-line, (16) reduces to

$$(17) \quad \Delta(\lambda) = \begin{vmatrix} 2a_1a_2a_3+\lambda & 2a_2^2a_3 & 2a_2a_3^2 \\ 2a_1^2a_3 & 2a_1a_2a_3+\lambda & 2a_1a_3 \\ 2a_1^2a_2 & 2a_2^2a_1 & 2a_1a_2a_3+\lambda \end{vmatrix}$$

If now the second cubic is taken in canonic form, instead of in the symbolic forms, symbolic products must be replaced as follows in $\Delta(\lambda)$

$\alpha_1 \alpha_2 \alpha_3$ by m, $\qquad \alpha_3^2 \alpha_1$ by c_1,

$\alpha_2^2 \alpha_3$ by b_3, $\qquad \alpha_1^2 \alpha_3$ by a_2

$\alpha_1 \alpha_3^2$ by c_2, $\qquad \alpha_2^2 \alpha_1$ by b_1.

$\alpha_1^2 \alpha_2$ by a_3

With these substitutions

$$(18) \quad f(\lambda) = \begin{vmatrix} m+\lambda & a_3 & a_2 \\ b_3 & m+\lambda & b_1 \\ c_2 & c_1 & m+\lambda \end{vmatrix}$$

$$(19) \quad = \lambda^3 + J_1 \lambda^2 + J_2 \lambda + J_3 , \quad \text{where}$$

$J_1 = 3m$.

$J_2 = 3m^2 - (b_1 c_1 + c_2 a_2 + a_3 b_3)$.

$$J_3 = \begin{vmatrix} m & a_3 & a_2 \\ b_3 & m & b_1 \\ c_2 & c_1 & m \end{vmatrix} = m^3 + a_3 b_1 c_2 + a_2 b_3 c_1 - m(b_1 c_1 + c_2 a_2 + a_3 b_3).$$

It is quite customary to write equations like (11) with different signs, but there appears to be no necessity, but an apparent disadvantage in so doing. In expressing T, it is necessary to use three components of

and also S in the form it assumes when in (5), we let $a = b = c = m$

i.e. $S = -(b_1^2 c_1^2 + c_2^2 a_2^2 + a_3^2 b_3^2) + c_2 a_2 a_3 b_3 + a_3 b_3 b_1 c_1 + b_1 c_1 c_2 a_2$.

To avoid we will use $I_1 = m$, instead of J_1. This is the I_1 of the present system so is doubly convenient.

Now take X, the Clebschian,

$$c_2 x_1^2 x_2 + b_1 x_1 x_3^2 + a_3 x_2^2 x_3 - c_1 x_1 x_2^2 - a_2 x_3^2 x_2 - b_3 x_1^2 x_3 = 0$$

and compare it with Salmons general form. First of all $a = b = c = m = 0$, and this at once causes T to assume the very simple form

$$(10).\ T = -6\, b_1 c_1 c_2 a_2 a_3 b_3 + [illegible](a_2^3 c_2^3 + [illegible]^3 + a_3^3 b_3^3) - [illegible](a_1^2 b_1^2 c_1^2 + [illegible] b_1^2 c_2^2)$$

$$12 (b_1^2 c_1^2 c_2 a_2 + [illegible] a_3 b_3 + [illegible] a_3 b_3 + [illegible] c_1 + [illegible] + [illegible] b_3 [illegible] a_2)$$

Comparing further we find we must replace

a_2 by c_2, $\quad a_3$ by $-b_3$

b_3 by a_3, $\quad b_1$ by c_1

c_1 by b_1, $\quad c_2$ by $-a_2$.

Making these subst. in T [illegible] (9) becomes

(21) $T = 6\, b_1c_1c_2a_2a_3b_3 - 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3) - 27(a_3^2b_1^2c_2^2 + a_2^2b_3^2c_1^2)$

$+ 12(b_1^2c_1^2a_2c_2 + b_1c_1^2a_2b_2 + c_2^2a_2^2a_3b_3 + c_2a_2^2b_1c_1 + a_3^2b_3^2c_1c_1 + a_3b_3^2c_2a_2)$.

which we may for brevity write

$T = 6x - 8y - 27z + 12t$.

We can now express T in terms of I_1, J_2, J_3 and S. Without going into the logic, the following combinations suggest themselves,

$(3I_1^2 - J_2)^3 = \sum^3 b_1^3c_1^3 + 3\sum^6 b_1c_1^2a_2c_2^2 + 6\, b_1c_1c_2a_2a_3b_3 = 6x + y + 3t$

$(J_3 + 2I_1^3 - I_1J_2)^2 = a_2^2b_1^2c_1^2 + a_3^2b_1^2c_2^2 + 2\, b_1c_1c_2a_2a_3b \quad = x + z$

$S(3I_1^2 - J_2) = -\sum^3 b_1c_1^3 + 3\, b_1c_1c_2a_2a_3b_3 \quad = 3x + y$,

and from these T is found at once by a proper choice of coefficients. We have that

(α) $T = 4(3I_1^2 - J_2)^3 - 27(J_3 + 2I_1^3 - I_1J_2)^2 + 12\, S(3I_1^2 - J_2)$.

Expanding terms and collecting

(β). $T = 54\, I_1J_2J_3 + 9I_1^2J_2^2 - 27J_3^2 - 4J_2^3 - 108\, I_1^3J_3 + 12\, S(3I_1^2 - J_2)$.

Now J_2, J_3 and S must be expressed in the coefficients, which have already been derived. For example J_2

and J_3 are [illegible] I_2 and $-I_3$ respectively. Hence

$$J_2 = [illegible]\, I_1^2 + D_1^2 + 12 I_2$$

$$J_3 = D_1(4 I_2 - I_1 D_1),$$

$$S = 12 I_2(D_1^2 + I_1^2) - 24 I_1 D_1 D_2 - 36(I_2^2 - D_2^2) - (I_1^2 - D_1^2)^2 .$$

Substituting these in (22)

$$(24).\ T = 4(I_1^2 - D_1^2 + 12 I_2)^3 - 27(4 D_1 I_2 - 12 I_1 I_2)^2$$
$$+ 12(12 I_2 D_1^2 + 12 I_2 I_1^2 - 24 I_1 D_1 D_2 - 36 I_2^2 + 36 D_2^2$$
$$- I_1^4 + 2 I_1^2 D_1^2 - D_1^4)(I_1^2 - D_1^2 + 12 I_2)$$

Expanding and collecting terms

$$(25)\ T = -8 I_1^6 + 8 D_1^6 + 24 I_1^4 D_1^2 - 24 I_1^2 D_1^4 - 1728 I_2^3 - 144 I_1 I_2$$
$$- 864 I_1^2 I_2^2 - 144 D_1^4 I_2 + 432 D_1^2 I_2^2 - 288 I_1^3 D_1 D_2$$
$$+ 288 I_1 D_1^3 D_2 + 432 I_1^2 D_2^2 - 864 D_1^2 D_2^2 - 864 I_1 I_2 D_1 D_2 + 5184 I_2 D_2^2 .$$

Or grouping terms, $T =$

$$(26)\ -8\Big[(I_1^2 - D_1^2)^3 - 216 I_2^3 - 18 I_2(I_1^4 - D_1^4) + 108(I_1^2 I_2^2 + D_1^2 D_2^2)$$
$$- 54(D_1 I_2 - I_1 D_2)^2 + 36 I_1 D_1 D_2 (I_1^2 - D_1^2) - 648 I_2 D_2^2\Big].$$

Now as S [illegible] necessarily [illegible], so must T be, if the result is correct. Carrying out the formation of the [illegible]

$$T' = (I_1^2 D_1^2 - 12 I_1 D_1 D_2 + 36 D_2 - D_1^4)^3 - 216 (D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2)^3$$

$$-18 (D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2)(I_1^4 D_1^4 - 24 I_1^3 D_1^3 D_2 + 216 I_1^2 D_1^2 D_2^2$$

$-864 I_1$ [illegible] $^3 + 12$ [illegible] $6 D_2^4$ [illegible] $D_1)$

$+108 \Big[(I_1^2 D_1^2 - 12 I_1 D_1 D_2 + 36 D_2^2)(D_1^2 I_2 -$ [illegible] $I_1 D_1 D_2 + 6 D_2^2)^2 -$ [illegible] $D_1 D_2$ [illegible] $\Big]$

$$-54 \left[D_1^2 (D_1^2 I_2 - 2 I_1 D_1 D_2 + 6 D_2^2) + D_1^2 D_2 (I_1 D_1 - 6 D_2) \right]^2$$

$$-36 D_1^4 D_2 (I_1 D_1 - 6 D_2)(I_1^2 D_1^2 - 12 I_1 D_1 D_2 + 36 D_2^2 - D_1^4)$$

$-$ [illegible] $D_1^4 I_2^2 (D_1 I_2$ [illegible] $I_1 D_1 D_2 + 6$ [illegible] $)$,

Expanding and collecting

$$= I_1^6 D_1^6 - D_1^{12} - 3 I_1^4 D_1^8 - 54 I_1^2 D_1^6 D_2^2 + 3 I_1^2 D_1^{10} - 36 I_1 D_1^9 D_2$$

$+10$ [illegible] $D_2^2 +$ [illegible] $D_1^7 D_2$ [illegible] $D_1^6 I_2^3$ [illegible] I_1^4 [illegible] D_1^6 [illegible] I [illegible] D_1^{10}

$$-108 I_1^2 I_2^2 D_1^6 - 54 D_1^8 I_2^2 - 648 I_2 D_1^6 D_2^2 + 108 I_1 I_2 D_1^7 D_2 .$$

$$= D_1^6 T.$$

So T is [illegible], being [illegible] of the [illegible] T

Knowing S and T we can immediately write down the discriminant of the Clebschian in terms of our invariants, since it is merely $T^2 + 64 S^3$. [illegible] the general case of two triangles in

[illegible]

[illegible] construction. If [illegible] the two triangles are [illegible] in-scribed in a conic, the form is much simplified for then $D_2 = 0$. Calling the discriminant D,

$$D = 6412\, I_2^2 D_1^2 \left(5 I_1^4 D_1^2 - 324 I_1^2 I_2^2 - 3 I_1^6 - I_1^2 D_1^4 + 521 I_1^4 I_2 + 48 I_1^2 I_2 D_1^2 - D_1^6 + 26 I_2 D_1^4 - 225 D_1^2 I_2^2 + 648 I_2^3\right).$$

Before taking up X', the line element, and its invariants, which really present little of importance, we will look for the invariant con-dition [illegible] the two triangles [illegible] [illegible].

Taking any triangle as [illegible] [illegible] [illegible] [illegible] is then as we had in equation (4)

(27). $\lambda = (a_2 b_3 c_3 + a_3 b_2 c_2 + a_2 b_2 c_3)\, x_1 x_2 + (a_1 b_1 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1)\, x_1 x_3^2$
$+ (a_3 b_1 c_1 + a_1 b_3 c_1 + a_1 b_1 c_3)\, x_2^2 x_3 - (a_1 b_3 c_3 + a_3 b_1 c_3 + a_3 b_3 c_1)\, x_1 x_2^2$
$- (a_2 b_1 c_1 + a_1 b_2 c_1 + a_1 b_1 c_2)\, x_2 x_3^2 - (a_3 b_2 c_2 + a_2 b_3 c_2 + a_2 b_2 c_3)\, x_1^2 x_3 .$

If we now take my triangles as 3-lines, that is write the point equation, the only effect on the coefficients in the second transvectant, λ', will be to replace each letter, by its minor in D_1. Of course the coordinates will be line coordinates, and the given triangle is referred to as reference triangle. Representing the minors by the corresponding Capitals we have

(28) $\lambda' = (A_2 B_3 C_3 + A_3 B_2 C_2 + A_2 B_2 C_3)\, \xi_1 \xi_2 + (A_1 B_1 C_2 + A_2 B_1 C_2 + A_1 B_2 C_1)\, \xi_1 \xi_3^2$
$+ (A_3 B_1 C_1 + A_1 B_3 C_1 + A_1 B_1 C_3)\, \xi_2^2 \xi_3 - (A_1 B_3 C_3 + A_3 B_1 C_3 + A_3 B_3 C_1)\, \xi_1 \xi_2^2$
$- (A_2 B_1 C_1 + A_1 B_2 C_1 + A_1 B_1 C_2)\, \xi_2 \xi_3^2 - (A_3 B_2 C_2 + A_2 B_3 C_2 + A_2 B_2 C_3)\, \xi_1^2 \xi_3 .$

Now to find the apolarity condition we will follow the usual procedure, regard the ξ's as differential operators, and equate the result of the operation to zero. This gives

$$29). \quad (a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)(A_2B_3C_3 + A_3B_2C_3 + A_3B_3C_2)$$
$$+ (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)(A_1B_2C_2 + A_2B_1C_2 + A_2B_2C_1)$$
$$+ (a_3b_1c_1 + a_1b_3c_1 + a_1b_1c_3)(A_3B_1C_1 + A_1B_3C_1 + A_1B_1C_3)$$
$$+ (a_1b_3c_3 + a_3b_1c_3 + a_3b_3c_1)(A_1B_3C_3 + A_3B_1C_3 + A_3B_3C_1)$$
$$+ (a_2b_1c_1 + a_1b_2c_1 + a_1b_1c_2)(A_2B_1C_1 + A_1B_2C_1 + A_1B_1C_2)$$
$$+ (a_3b_2c_2 + a_2b_3c_2 + a_2b_2c_3)(A_3B_2C_2 + A_2B_3C_2 + A_2B_2C_3) = 0.$$

The work of expanding (29) is long and the result only will be indicated. It requires the summing of 6 terms like

$$(a_2b_3c_3 + a_3b_2c_3 + a_3b_3c_2)(2a_1^2b_1b_2c_2c_3 + 2a_2a_3b_1^2c_1c_2 + 2a_1a_2b_2b_3c_1^2$$
$$-2a_1^2b_2b_3c_1c_2 - 2a_2a_3b_1b_2c_1^2 - 2a_1a_2b_1^2c_2c_3 + a_2^2b_1^2c_1c_3 + a_1a_3b_2^2c_1^2$$
$$+ a_1^2b_1b_3c_2^2 - a_2^2b_1b_3c_1^2 - a_1a_3b_1^2c_2^2 - a_1^2b_2^2c_1c_3).$$

[illegible] very concisely thus

$$(30) \quad 2\Big(\sum^{9} a_1a_2a_3b_2^2b_3c_1^2c_3 - \sum^{9} a_1a_2a_3b_1^2b_3c_2^2c_3\Big)$$
$$+ 3\Big(\sum^{6} a_1a_2^2b_2b_3^2c_3c_1^2 - \sum^{6} a_1a_2^2b_3b_1^2c_2c_3^2\Big)$$
$$+ 2\Big(\sum^{9} a_1^3b_2b_3^2c_2^2c_3 - \sum^{9} a_1b_2^2b_3c_2c_3^2\Big) = 0$$

To express this in terms of the fundamental invariants, we observe first of all that the C's [illegible]

terms to have opposite [illegible]. We see then that D_1^3, $I_1^2D_1$, I_1D_2 and I_2D_1 are the only terms which can enter. Determining the coefficients we have finally

(31) $D_1^3 - I_1^2D_1 - 3I_1D_2 + 9I_2D_1 = 0.$

Hence the theorem:

The two Clebschians of two triangles are apolar if

$$D_1^3 - I_1^2D_1 - 3I_1D_2 + 9I_2D_1 = 0$$

This expression is also self dual as we should expect.

We [illegible] need to [illegible] the invariants S' and T' of χ', that is of the Clebschian arising from the line equation. The equation of this line cubic has already been given (28). Hence the coefficients [illegible] the "dual" of the coefficients of χ, (27) it is evident that we can [illegible]

invariant forms of the [illegible] of χ' by simply applying the dual transformation to the corresponding forms of χ. But we found that S and T were self dual forms. Hence

$$S' = S \quad \text{and} \quad T' = T$$

[illegible] as power of I_1. [illegible] and hence the theorem.

If neither of two triangles is degenerate, the same condition $[S = 0]$ which makes the Hessian of [illegible] Clebschian break down into three lines, makes the Hessian of the other Clebschian degenerate into three points.

A corresponding theorem could be formulated involving the fact that T and T' vanish simultaneously.

The fact that the two Clebschians in the case of two triangles do not coincide, can be verified by calculating the line equation of (27) which does not

[illegible] these [illegible] these [illegible] these two equations of the [illegible] but the truth of the statement [illegible] is evident from [illegible] the [illegible] of E, which [illegible]

57. Self-dual Invariant Forms.

A number of self-dual forms have arisen in the course of this article, and the question naturally arises as to the number of such forms which are independent. These self-dual forms acquire special interest from the fact that the vanishing of such a form indicates a mutual relation between the two triangles. We shall look for the most general forms of the first degree which are left unaltered by the dual transformation, to within a power of D_1.

Of degree one, the only form is

1) D_1.

For the form of degree two, the most general one is,

2) $aI_1^2 + bI_2 + cD_1 + dI_1D_1 + eD_1^2$. [illegible]

that this equal the dual forms, i.e.

$$D_1^2(aI_1^2 + bI_2 + cD_2 + dI_1D_1 + eD_1^2)$$

$$\equiv a(I_1^2D_1^2 - 12 I_1D_1D_2 + 36 D_2^2)$$

$$+ b(D_1^2I_2 - 2I_1D_1D_2 + 6D_2^2) - cD_1^2D_2$$

$$+ d\,D_1^2(I_1D_1 - 6D_2) + eD_1^4.$$

Equating coefficients, the equations of condition are

$$12a + 2b = 0 \qquad b = b$$

$$36a + 6b = 0$$

$$-c - 6d = +c$$

$\therefore$ $c = 3d$ + $b = -6a$ are the only restrictions. $\therefore$

$\therefore$ $a(I_1^2 - 6I_2) + d(I_1D_1 - 3D_2) + eD_1^2 = 0$, is the most general self-dual invariant form of the second degree, where a, d and e are arbitrary constants. Hence

(3). $I_1^2 - 6I_2$ and $I_1D_1 - 3D_2$ are the only two independent, irreducible self-dual invariants of degree 2. It may be noted that while $I_2 = 0$ expresses a

mutual relation, being one of these two mutual ones, it is not strictly self dual, for it changes sign under the dual transformation, hence could not be used in building up a pencil of self dual forms. D_2^2 is self dual.

For the most general dualistic form we must have

$$(4).\ D_1^3(aI_1^3 + bD_1^3 + cI_1^2D_1 + dI_1D_1^2 + eI_1I_2 + fI_1D_2 + gD_1D_2 + hI_2D_1 + 2kI_3)$$

$$\equiv a(I_1^3D_1^3 - 18I_1^2D_1^2D_2 + 108I_1D_1D_2^2 - 216D_2^3)$$

$$+ bD_1^6 + cD_1^2(I_1^2D_1^2 - 12I_1D_1D_2 + 36D_2^2)$$

$$+ d(I_1D_1^5 - 6D_1^4D_2)$$

$$+ e(I_1I_2D_1^3 - 2I_1^2I_1D_1 + 6I_1D_1D_2^2 - 6I_1I_1D_2 + 12I_1D_1D_2^2 - 36D_2^3)$$

$$+ f(-I_1D_1^3D_2 + 6D_1^2D_2^2) - g(D_1^4D_2)$$

$$+ h(D_1^4I_2 - 2I_1D_1^3D_2 + 6D_1^2D_2^2) + k(2D_1^3I_3 - 2D_2^3 + I_1D_1D_2^2 - I_2D_1^2D_2).$$

Since a number of the equations of condition resulting from the identity are equivalent, we need only write such as are distinct, taking one

$-18a - 2c = 0$ $\quad\therefore\ c = -9a$

$108a + 18c + h = 0$ $\quad\therefore\ h = 54a$

$36c + 6g + 6h = 0$ or $6c = -(g+h)$

$-6d - g = g$ $\quad$ or $g = -3d$

Introducing these equalities in the left of (4) we have

$a(I_1^3 - 9I_1I_2 + 108 I_3) + 2c(D_1$ [illegible]$)$

$+ c(I_1^2 D_1 - 6 I_1 D_2) + d(I_2 D_1 - I_1 D_2)$,

the most general self dual invariant cubic form. We have here three independent cubics which are self dual

I. $\quad I_1^3 - 9 I_1 I_2 + 108 I_3$.

II. $\quad I_1^2 D_1 - 6 I_1 D_2$.

III. $\quad I_2 D_1 - I_1 D_2$.

It is well to consider the 6 simple self dual invariants we have derived: They are not independent, as we might suspect. For we should reasonably expect not more than five independent ones as our entire rational

system consisted of but six. Of course, this does not mean that all self dual forms are expressible rationally in terms of five. A distinct set of five is

$$\text{A.}\quad\begin{cases} D_1\,; \\ I_1D_1 - 3D_2\,; \quad I_1^2 - 6I_2\,; \\ I_2D_1 - I_1D_2\,; \ \& \ I_1^3 - 9I_1I_2 + 108I_3\,. \end{cases}$$

The other cubic form on the last page can be expressed thus

$$I_1^2D_1 - 6I_1D_2 = 6(I_2D_1 - I_1D_2) + D_1(I_1^2 - 6I_2).$$

The most general self dual quartic invariant was found to be

$$a(I_1^4 - 12I_1^2I_2 + 36I_2^2) + b(I_1D_1I_2 - 12I_3D_1 - \tfrac{1}{9}I_1^3D_1)$$
$$+c(-4I_3D_1 - 6I_1D_2 \ [illegible]\ I_1 \ [illegible] + I_1 I_2) + d(I_1^2D_1 - 6I_1D_1I_2)$$
$$+e(D_1^2I_2 - I_1D_1D_2) + f(I_1D_1^3 - 3D_1^2D_2) + g\,D_2^2 + h\,D_1^4.$$

It turns out that all the several simple invariants making up the general form are expressible in terms of the 5 forms of (A). D_2^2 seems at first to be an independent quartic

form, but it can be written

$$D_2^2 = \frac{1}{9}\left[(I_1 D_1 - 3D_2)^2 - D_1(I_1^2 D_2 - 6 I_1 D_2)\right]$$

$$= \frac{1}{9}\left[(I_1 D_1 - 3D_2)^2 - 6D_1(I_1 D_1 \text{[illegible]}) \; D_1^2(I_1^2 - 6 I_2)\right].$$

This shows that it is of advantage to retain the third cubic form for certain purposes.

The most general self dual quintic has also been found, and while there was a theorem of invariant expressing in terms of the fundamental expressions, such one was ultimately expressible in terms of our four simple ones. The proof was there that if any invariant can be expressed in terms of these four it is self dual. The converse was not then generally proven, but we found all self dual forms we have found can be so expressed. For example from the previous chapter S and T fur [illegible]

[illegible]

really covered by the general quartic case.

$$S = 4(I_1D_1 - 3D_2)^2 - (I_1^2 + D_1^2 - 6I_2)^2$$

$$= 4(I_1D_1 - 3D_2)^2 - [(I_1^2 - 6I_2) + D_1^2]^2.$$

$T = (I_1^2 \; D_1^2)^3 \; 2 \cdot 6 I_1^2$ [illegible]

$54($ [illegible] $- I_1 I_2) + 36 I_1 I_2$ [illegible] $(I_1^2 D_1)$ [illegible].

$$= (I_1^2 - 6I_2)^3 - D_1^6 - 3(I_1^2D_1 - 6I_1D_2)^2 + 54(I_2D_1 - I_1D_2)^2$$
$$+ 3(36D_2^2 + D_1^4)(I_1^2 - 6I_2) + 36D_1^3(I_2D_1 - I_1D_2)$$
$$+ 108D_1^2D_2^2.$$

And since

$$4D_2^2 = (I_1D_1 - 3D_2)^2 - 6D_1(I_2D_1 - I_1D_2) - D_1^2(I_1^2 - 6I_2)$$

& $\quad I_1^2D_1 - 6I_1D_2 = 6(I_2D_1 - I_1D_2) + D_1(I_1^2 - 6I_2)$

we have in terms of [illegible]

$$T = (I_1^2 - 6I_2)^3 - D_1^6 - 162(I_2D_1 - I_1D_2)^2 - 15D_1^2(I_1^2 - 6I_2)^2$$
$$- 108(I_2D_1 - I_1D_2)D_1(I_1^2 - 6I_2) + 12(I_1D_1 - 3D_2)^2(I_1^2 - 6I_2)$$
$$- 36D_1^3(I_2D_1 - I_1D_2) + 12D_1^2(I_1D_1 - 3D_2)^2 - 9D_1^4(I_1^2 - 6I_2).$$

This might be grouped more compactly but it establishes the point aimed at. It thus turns out that all self dual forms

arising are found to be rational functions of these five.

If an invariant form is self dual it is naturally of peculiar interest. Unfortunately, however, the simpler self dual forms [illegible] treated in the body of this article, present no simple geometric interpretation.

$D_1 = H_1$

$I_1 D_1 - 3 D_2 = H_2 \qquad I_1^2 - 6 I_2 = \bar{K}_2$

$I_2 D_1 - I_1 D_2 = H_3 \qquad I_1^3 - 9 I_1 I_2 + 108 I_3 = \bar{K}_3$

$D_1 = H_1$

$-3D_2 = H_2 - I_1 H_1$

$-6 I_2 = \bar{K}_2 - I_1^2$

$108 I_3 = \bar{K}_3 - \frac{3}{2} I_1 \bar{K}_2 + \frac{1}{2} I_1^3$

From * $\quad I_1^2 H_1 - 2 H_2 I_1 = - H_3 + H_1 \bar{K}_2$

Hence any self dual invariant by multiplic[ation] with a power of H_1 can be changed into an invariant of the form $S_1 + I_1 S_2 \equiv I$ where S_1 and S_2 are functions of the self dual invariants $H_1\ H_2\ K_2\ H_3\ K_3$

Since I, S_1, S_2 and $S_1 + I_1 S_2$ are all self dual while I_1 is not, S_2 must vanish and I is a function of $H_1\ H_2\ K_2\ H_3\ K_3$ alo[ne]

Hence these five self-dual invariants are a complete system of self dual invariants

§10. The Fundamental Invariants under a Special Cremona Transformation.

It is the purpose here to show the changes produced in the fundamental invariants, by carrying out a special Cremona transformation, and to illustrate by a single example the value of this knowledge.

Take the simple, quadratic involutory transformation

$$y_i = \frac{k}{x_i}.$$

The effect of this transformation is to replace each symbol in the invariant by its reciprocal. Since all invariant forms are homogeneous, it is clear that the same power of k appears in each term, hence can be dropped without loss of generality.

Carrying out this transformation on the fundamental invariants

$$B.\left\{\begin{array}{ll} D_1' = \frac{D_2}{I_3} & I_2 = \frac{I_1}{I_3} \\ D_2' = \frac{D_1}{I_3} & I_3 = \frac{1}{I_3} . \\ I_1 = \frac{I_2}{I_3} & \end{array}\right.$$

As a typical illustration we take the rational quintic which we can write parametrically as

$$x_i = \frac{t - x_i}{(t-\beta_i)^2}$$

which has 3 cusps, 3 double points, 3 double tangents, 3 flexes and is of class 5, and which has been the subject of considerable study. ~~It~~ It can be shown by the simple counting of constants that any two of the singular triangles are not independent, but there is one general relation connecting them. Hence taking the triangle of double points as our reference [illegible] we could

ask; What is the condition on the triangle of cusps?

From the theory of Cremona transformations,* it is well known that if the above quadratic transformation is carried out on the quintic given, taking the double points as fundamental points, the quintic goes into a rational quartic with three cusps, and having simple points at the three fundamental points. If we now carry out the same transformation regarding the cusps as fundamental points (which is itself equivalent to taking the dual) the rational quartic goes into a conic, which touches the fundamental lines, and has the three cusp points for ordinary points. But we already

* Clebsch - Leçons - Vol II, page 192 ff.

Cremona - Loc cit.

[illegible] chapter the condition that a conic may be inscribed in the 3-line and circumscribed about the 3-point. It is

$$(1).\ (D_1^4 - 8D_1^2 I_2 + 16 D_2^2)^2 - 512 I_3 D_1^3 (I_1 D_1 - 4D_2) = 0.$$

Hence to find the condition on the cusp triangle we must take (1) and carry out in reverse order the transformations we [illegible] on the curve. That is first carry out transformation (B) of this section, then the dual transformation of [illegible] chapter, and finally transformation (B), again. This gives a rather long expression of the 24th degree in the coefficients of the original triangles. So the relation is probably not a very simple one geometrically.

If [illegible], the cusp [illegible] is [illegible] the [illegible]

3-line; the condition on the 3-point of double point becomes one of the 18th degree. By a similar argument assuming these forms

$$D_1^8 D_2^5 + (16 I_1^2 D_1^2 D_2 - 64 I_1 D_1 D_2^2 + 64 D_2^3 - 16 D_1^4 D_2 - 256 I_3 D_1^3)$$
$$(2 D_1^3 I_3 - 2 D_2^3 + I_1 D_1 D_2^2 - I_2 D_1^2 D_2)^2$$
$$- D_2 (16 I_2 D_1^6 D_2 - 24 I_1 D_1^5 D_2^2 + 4 D_1^4 D_2^3)(2 D_1 I_3 - 2 D_2^3 + I_1 D_1 D_2^2 - I_2 D_1 D_2) = 0.$$

This is sufficient to illustrate our point. For by a direct attack this problem would have been impossible, yet by aid of this transformation it is comparatively simple. Other examples might be carried out to still further illustrate this.

Incidentally this gives me a convenient geometrical interpretation of $I_2 = 0$. For if we carry out the transformation $y_i = \frac{1}{x_i}$ on the coordinates of my 3-point they are carried into three new points whose equations are

$$a_2 a_3 \xi_1 + a_3 a_1 \xi_2 + a_1 a_2 \xi_3 = 0$$

$$b_2 b_3 \xi_1 + b_3 b_1 \xi_2 + b_1 b_2 \xi_3 = 0$$

$$c_2 c_3 \xi_1 + c_3 c_1 \xi_2 + c_1 c_2 \xi_3 = 0$$

and clearly the conditions that these be apolar to the reference 3-line is $I_1 = 0$. Hence we a definite theorem:—

If two triangles are such that when one is taken as a three-line and the other as a three point, and the transformation $\rho_i = \frac{1}{\xi_i}$ is carried out, the three point is apolar to the three line, then

$$I_2 = 0.$$

Of course this gives no direct information as to the geometrical relation between the two triangles in their original position.

Vita.

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vanced courses under Professors Morley, Cohen, Coble, Whitehead and Reid. To each of these he would express his thanks for the kindness and interest shown at all times, especially to Dr Coble for his many valuable suggestions and to Professor Morley for his constant inspiration during the preparation of this dissertation, and throughout his course.